Truth as a logical connective

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> I swear, in the kingdom of generalities, you could be imperius rex.
> (Haruki Murakami "A Wild Sheep Chase")

1 Introduction

It is known that many truth theories like **FS**, which allow to represent "infinite conjunctions" in the sense of deflationism, are ω-inconsistent. In this paper, we examine these aspects by means of regarding the truth as a logical connective in terms of proof theoretic semantics. We do not try to assert that the truth conception should be represented by a logical connective instead of the predicate philosophically: only we want to do is to provide a new perspective to analyze the problem of ω-consistency which is caused by McGee's paradox, Yablo paradox and so on.

The object of the analysis of this paper is Freidman-Sheared's truth theory **FS** which is known to be ω-inconsistent. It consists of all axioms and schemata of **PA**, and the following special rules:

- Formal Commutatibity: for any logical connective $\circ$ and quantifier Q,

$$\mathbf{Tr}(x \dot{\circ} y) \equiv \mathbf{Tr}(x) \circ \mathbf{Tr}(y) \qquad \mathbf{Tr}(Qz(x(z))) \equiv Qz\mathbf{Tr}(x(z))$$

- two inference rules $\mathbf{NEC}, \mathbf{CONEC}$ as follows:

$$\frac{\varphi}{\mathbf{Tr}(\varphi)}\ \mathbf{NEC} \qquad \frac{\mathbf{Tr}(\varphi)}{\varphi}\ \mathbf{CONEC}$$

It is natural to ask the following question:

> **Question:** Is it possible to think the truth predicate **Tr** of **FS** as a **logical connective**?

It is because **NEC** is the introduction rule of the truth predicate and **CONEC** is the elimination rule of that themselves. From a nave proof theoretic semantics viewpoint which insists that the meaning of a logical connective is decided by the introduction rule and the elimination rule, it seems no problem to call **Tr** a logical connective.

However, there is a problem to call so: $\mathbf{Tr}(x)$ cannot be a logical connective because it violates the **HARMONY** between **NEC** and **CONEC**. Harmony is a central principle of the proof theoretic semantics which requires the appropriate relationship between the introduction rule and the elimination rule, and it guarantees various preservation results, in particular the normalizability of proofs. The violation of the harmony involves some unexpected results: the ω-inconsistency of **FS** in particular. It is said that this is one of the most serious defects of **FS**.

In this paper, we analyze this problem in terms of computer science. ω-inconsistency is due to the fact that potentially infinite (or *coinductive*) formulae are definable by finitely method in **FS**. In McGee's paradox case, the paradoxical sentence γ is intuitively of the form $\neg\mathbf{Tr}(\mathbf{Tr}(\mathbf{Tr}(\cdots)))$. Each step of its delivation, like $\mathbf{Tr}(\mathbf{Tr}(\cdots)))$ implies $\mathbf{Tr}(\mathbf{Tr}(\mathbf{Tr}(\cdots))))$, is finitely and computable, but the stream is of infinite length and therefore not computable as a whole. Actually the definability of such infinite streams are necessary consequence of deflationism, which allows the sentence whose intuitive interpretation is an infinite conjunction of sentences. As for the violation of the harmony, traditional harmony works for inductively defined formulae effectively, but not for coinductive formulae: it is clear that the normalization in *finite* steps is impossible for such infinite formulae. In particular, the formal commutativity is more problematic than other rules because it can be regarded as an infinitely operation though other inference rules are finitely.

As a consequence, therefore, if we want to call **Tr** a logical connective, we should extend the concept of harmony appropriately. The way of such extension is already known in computer science: we introduce the solution of this problem by using the concept of *guarded correcursion* along the line of [**?**], and we will show the positive answer. That is, according the extended concept, if we abandon the formal commutatibity, then we can regard that **Tr** with **NEC**, **CONEC** rule is a logical connective.

These potentially infinite sentences are not mere artifacts. Often we see a statement that its single inference step is valid and verifiable but it is not grounded, i.e. we cannot achieve its ground effectively like:

"Who said such thing?"
"Taro said so."
"Why did he say so?"
"Taro heard Jiro said so"
"Why did he say so?"
" Jiro heard Saburo said so"

and so on. Though this grounding process will terminate eventually since there are finitely many humans on earth, but with respect to effectively, such process are represented by coinductive objects in the tradition of computer science because nobody can brave out this process and achieve the final ground. The nature of truth, if any, in the context of this paper is that the truth conception of deflationism makes it possible to define such "coinductive" grounding process by using the technique of infinite conjunction. And this is a common cause of the McGee's paradox, Yablo paradox and so on.

Many people believed that only sentences of finite length should be taken seriously in the context of truth theory. However, the consequence of this paper is opposite: we should take such coinductive nature of truth seriously.

2 Preliminaries

2.1 Liar paradox and Friedman-Sheared Truth theory FS

Tarskian T-schemata (unrestricted form): for any formula φ,

$$\varphi \equiv \mathbf{Tr}(\ulcorner\varphi\urcorner)$$

where $\ulcorner\varphi\urcorner$ is a Godel code of φ.

"Snow is white." is true iff snow is white.

The fixed point lemma: For any $P(x)$, there is a closed formula ψ such that $\psi \equiv P(\ulcorner\psi\urcorner)$.
the intuitive meqaning of ψ: $P(\ulcorner P(\ulcorner P(\ulcorner \cdots \urcorner)\urcorner)\urcorner)$

The liar paradox

$$\Lambda \equiv \neg\mathbf{Tr}(\ulcorner\Lambda\urcorner)$$

Two solutions

The solution 1: keep classical logic

abandan the full form of T-scheme (to exclude the liar sentence) [Ha11]

e.g. Friedman-Sheared's **FS**

$$\frac{\varphi}{\mathbf{Tr}(\varphi)}\ \mathbf{NEC} \qquad \frac{\mathbf{Tr}(\varphi)}{\varphi}\ \mathbf{CONEC}$$

- For any φ, T-sentence is provable
- But the T-sentence of the liar is a theorem of **FS**.

Such restrictions **prevents** that the theory implies the liar sentence as a theorem.
to restrict T-scheme not to prove Λ, certain property (McGee [M85]) as

$$\mathbf{Tr}(\ulcorner A \to B\urcorner) \not\equiv \mathbf{Tr}(\ulcorner A\urcorner) \to \mathbf{Tr}(\ulcorner B\urcorner)$$

The solution 2: kepp the unrestricted form of T-schemata

- and abandon classical logic
  - paracomplete logic ([Fl08])
  - paraconsistent logic ([B08])
  - fuzzy logic ([**?**][R93]) , etc.

2.2 Friedman-Sheared's Axiomatic truth theory FS

FS = **PA** + the **formal commutability** of **Tr**
+ the **introduction** and the **elimination rule** of **Tr**

Definition 2.1 *Friedman-Sheared's Axiomatic truth theory* **FS***consists of the following axioms:*

- *Axioms and schemes of* **PA** *(mathematical induction for all formulae including* **Tr***)*
- *The* **formal commutability** *of the truth predicate:*
  - *logical connectives*
    * *for amy atomic formula ψ,* $\mathbf{Tr}(\ulcorner\psi\urcorner) \equiv \psi$,
    * $(\forall x \in \mathbf{Form})[\mathbf{Tr}(\dot\neg x) \equiv \neg\mathbf{Tr}(x)]$,
    * $(\forall x, y \in \mathbf{Form})[\mathbf{Tr}(x \dot\wedge y) \equiv \mathbf{Tr}(x) \wedge \mathbf{Tr}(y)]$,
    * $(\forall x, y \in \mathbf{Form})[\mathbf{Tr}(x \dot\vee y) \equiv \neg\mathbf{Tr}(x) \vee \mathbf{Tr}(y)]$,
    * $(\forall x, y \in \mathbf{Form})[\mathbf{Tr}(x \dot\to y) \equiv \neg\mathbf{Tr}(x) \to \mathbf{Tr}(y)]$,
  - *quantifiers*
    * $(\forall x \in \mathbf{Form})[\mathbf{Tr}(\dot\forall z\, x(z)) \equiv \forall z \mathbf{Tr}(x(z))]$,
    * $(\forall x \in \mathbf{Form})[\mathbf{Tr}(\dot\exists z\, x(z)) \equiv \exists z \mathbf{Tr}(x(z))]$,

- *The* **introduction rule** *and the* **elimination rule** *of* $\mathbf{Tr}(x)$

$$\frac{\varphi}{\mathbf{Tr}(\ulcorner\varphi\urcorner)}\ \mathbf{NEC} \qquad \frac{\mathbf{Tr}(\ulcorner\varphi\urcorner)}{\varphi}\ \mathbf{CONEC}$$

where **Form** *is a set of Godel code of formulae.*

2.3 PTS of Tr

A proof theoretic semantics (PTS) of **Tr** (naive version) **FS**'s two rules look like the introduction and the elimination rule; **Tr** looks like a logical connective[Hj12]

$$\frac{\varphi}{\mathbf{Tr}(\varphi)}\ \mathbf{NEC} \qquad \frac{\mathbf{Tr}(\varphi)}{\varphi}\ \mathbf{CONEC}$$

"the meaning of word is its use in the language" (Wittgenstein)

A (naive) proof theoretic semantics: **the meaning of a logical connective is given by the introduction rule and the elimination rule**

To regard **Tr** as a logical connective involves the following: the essence of **Tr** is that we can trace the argument; the **justification of the consequence** can be **reduced** to the **justification of the assumptions** even though **Tr** is used in the argument

(the **the reducibility of the justifiability**). We can reduce the argument in which we use **Tr** to that which does not contain **Tr** by **finite steps**.

2.4 Motivations

2.4.1 1st. motivation: Deflationism

Regarding the truth predicate as a logical connective is important from truth theoretic viewpoint:

Deflationism= Quinean disquotational view of truth + the commutability to quantifier

"disquotation view of truth" (Quine, etc.): The role of truth predicate seems to "**disquote**" quoted sentences $\ulcorner\varphi\urcorner$ (then we get φ). According to them, truth does not have a signigficant role in semantics.

Deflationists allows to assert "**infinite conjunctions of sentences**" by using truth predicate:

e.g. " Everything he said is true"

Formally: Assume $\varphi_0, \varphi_1, \cdots$ is a recursive enumeration of sentences, and f is a recursive function s.t. $f(n) = \ulcorner\varphi_n\urcorner$. Then the following represents the infinite conjunction of $\varphi_0 \wedge \varphi_1 \wedge \cdots$

$$\forall x \mathbf{Tr}(f(x))$$

% ontological commitment 以外の話を出す！

Deflationists insist that truth's influence to the semantics should be limited: Syntax theory Connection to this talk: Logical connectives are typical examples which do not involve ontological assertions

2.4.2 2nd. motivation: Proof theoretic semantics(PTS)

A (naive) proof theoretic semantics (PTS):
the meaning of a logical connective is given by the introduction rule and the elimination rule

"the meaning of word is its use in the language" (Wittgenstein)

Some PTS people pointed out that a truth predicate with NEC, CONEC can be regarded as a logical connective[Hj12].

To regard **Tr** as a logical connective involves the following:

- the essence of logical connectives is that the **justification of the consequence** can be **reduced** to the **justification of the assumptions** even though **Tr** is used in the argumen.t

  (the **the reducibility of the justifiability**).

- So is the essence of **Tr**. In particular, we can reduce the argument in which we use **Tr** to that which does not contain **Tr** by **finite steps**.

2.4.3 Another purpose

Truth theory without Arithmetization!

Analyzing the formalized conceptions of truth from computer scientific viewpoint... Arithmetic is regarded as a theory of syntax, Arithmetization is not essential for the study of conception of truth, but ...

The problems of arithmetization, like ω-inconsistency of **FS**, are now in the center of the investigation, but non-standard arithmetic is very difficult to study.

Truth theory without Arithmetization!
For example, in Type Theory

- Inductively defined formulae are members of a **inductive data type** of formulae,
- This is more intuitive than nonstandard arithmetic!

3 A problem: the violation of HARMONY

Tr(x) **in FS cannot be a logical connective**
because **it violates the "HARMONY"** between the introduction rule and the elimination rule.

3.1 HARMONY

What is "HARMONY"?

To say "**Tr** is a logical connective", it must satisfy **the HARMONY between the introduction rule and the elimination rule**.

An example (TONK) The following connective without HARMONY trivializes the system.

$$\frac{A}{A\,\mathbf{tonk}\,B}\ \mathbf{tonk}+ \qquad\qquad \frac{A\,\mathbf{tonk}\,B}{B}\ \mathbf{tonk}-$$

There are some criteria of "HARMONY":

- **Concervative extension** (Belnap, Dummett)

  For any φ which does not include **Tr**, if **FS** proves φ, then **PA** proves it. Conservative extension criterion corresponds to deflationism of truth.

- Normalizability of proofs (Dummett)

  Any proof can be rewritten to its normal form.

The violation of "HARMONY"
However, **Tr** does **NOT** satisfy the criteria of **HARMONY**:

- **FS** proves the consistency of **PA**: adding **NEC, CONEC** is not a conservative extension!
- In particular, McGee's: **FS** is ω-inconsistent [M85],

  Actually NEC and CONEC make the number of provable sentences drastically large!

- Yablo's paradox: Truth theories in classical logic wth enough-strong expressive power is either
  - contradictory; it is provable without using self-referential sentences directly [Yb93],
  - or it is ω-inconsistent [L01]

What should we do if we want to say **Tr** is a logical connective?

3.2 McGee's Theorem

Theorem[McGee] Any consistent truth theory T which include **PA** and **NEC** and satisfies the following:

(1) $(\forall x, y)[x, y \in \mathbf{Form} \to (\mathbf{Tr}(x \dot{\to} y) \to (\mathbf{Tr}(x) \to \mathbf{Tr}(y)))]$,

(2) $\mathbf{Tr}(\bot) \to \bot$,

(3) $(\forall x)[x \in \mathbf{Form} \to \mathbf{Tr}(\dot{\forall} y x(y)) \to (\forall y \mathbf{Tr}(x(y)))]$

Then T is ω-inconsistent

proof We define the following paradoxical sentence

$$\gamma \equiv \neg \forall x \mathbf{Tr}(g(x, \ulcorner \gamma \urcorner))$$

where g is a recursive function s.t.

$$\begin{aligned} g(0, \ulcorner \varphi \urcorner) &= \ulcorner \mathbf{Tr}(\ulcorner \varphi \urcorner) \urcorner \\ g(x+1, \ulcorner \varphi \urcorner) &= \ulcorner \mathbf{Tr}(g(x, \ulcorner \varphi \urcorner)) \urcorner) \end{aligned}$$

The intuitive meaning of γ (I don't know how many **Tr**'s!)

$$\gamma \equiv \neg \mathbf{Tr}(\ulcorner \mathbf{Tr}(\ulcorner \mathbf{Tr}(\ulcorner \mathbf{Tr}(\ulcorner \cdots \urcorner) \urcorner) \urcorner) \urcorner)$$

Then we can prove γ: roughly

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\gamma \equiv \neg\mathbf{Tr}(\ulcorner\mathbf{Tr}(\ulcorner\mathbf{Tr}(\ulcorner\mathbf{Tr}(\ulcorner\cdots\urcorner)\urcorner)\urcorner)\urcorner)}{\gamma \to \neg\mathbf{Tr}(\ulcorner\mathbf{Tr}(\ulcorner\mathbf{Tr}(\ulcorner\mathbf{Tr}(\ulcorner\cdots\urcorner)\urcorner)\urcorner)\urcorner)}}{\mathbf{Tr}(\ulcorner\gamma \to \neg\mathbf{Tr}(\ulcorner\mathbf{Tr}(\ulcorner\mathbf{Tr}(\ulcorner\mathbf{Tr}(\ulcorner\cdots\urcorner)\urcorner)\urcorner)\urcorner)\urcorner)}}{\mathbf{Tr}(\ulcorner\gamma\urcorner) \to \mathbf{Tr}(\ulcorner\neg\mathbf{Tr}(\ulcorner\mathbf{Tr}(\ulcorner\mathbf{Tr}(\ulcorner\mathbf{Tr}(\ulcorner\cdots\urcorner)\urcorner)\urcorner)\urcorner)\urcorner)}\ FC}{\mathbf{Tr}(\ulcorner\gamma\urcorner) \to \neg\mathbf{Tr}(\ulcorner\mathbf{Tr}(\ulcorner\mathbf{Tr}(\ulcorner\mathbf{Tr}(\ulcorner\mathbf{Tr}(\ulcorner\cdots\urcorner)\urcorner)\urcorner)\urcorner)\urcorner)}\ FC}{\mathbf{Tr}(\ulcorner\gamma\urcorner) \to \gamma}$$

And

$$\dfrac{\dfrac{\neg\gamma \to \forall x\mathbf{Tr}(g(x, \ulcorner\gamma\urcorner))}{\neg\gamma \to \mathbf{Tr}(g(\bar{0}, \ulcorner\gamma\urcorner))}}{\neg\gamma \to \mathbf{Tr}(\ulcorner\gamma\urcorner)}$$

Therefore $\mathbf{Tr}(\ulcorner\gamma\urcorner) \to \gamma$ and $\neg\gamma \to \mathbf{Tr}(\ulcorner\gamma\urcorner)$ implies, $\mathbf{FS} \vdash \neg\neg\gamma$ though $\neg\mathbf{Tr}(g(\bar{n}, \ulcorner\gamma\urcorner))$ is provable for any n.

This is ω-inconsistency:

- Since γ is provable, all $\mathbf{Tr}(g(\bar{0}, \ulcorner\gamma\urcorner)), \mathbf{Tr}(g(\bar{1}, \ulcorner\gamma\urcorner)), \mathbf{Tr}(g(\bar{2}, \ulcorner\gamma\urcorner)), \cdots$ are provable. Therefore, for any natural number n, $\mathbf{Tr}(g(\bar{n}, \ulcorner\gamma\urcorner))$ is provable.
- However, γ, i.e. $\neg\forall x\mathbf{Tr}(g(x, \ulcorner\gamma\urcorner))$ is provable.

Model theoretically speaking, there is a non-standard natural number d such that $\neg\mathbf{Tr}(g(d, \ulcorner\gamma\urcorner))$ in any model of **FS**. Intuitively it is of the form $\neg\underbrace{\mathbf{Tr}(\ulcorner\mathbf{Tr}(\ulcorner\mathbf{Tr}(\ulcorner\mathbf{Tr}(\ulcorner\cdots\urcorner)\urcorner)\urcorner)\urcorner)}_{\text{length } d}$.

In formal arithmetics, natural numbers are usually represented **terms** which are the combination of the following symbols:

- constant symbol: $\bar{0}$
- function symbol $S(x)$ (successor function)

Therefore, the natural number 0 is represented by the term $\bar{0}$, 1 is represented by $S(\bar{0})$, 2 is by $S(S(\bar{0}))$, $\cdots$ Two kinds of natural numbers in models of formal arithmetics:

- **standard** natural numbers:
  They are "finite" natural numbers, i.e. it can be written by digits in principle, $0, 1, 2, \cdots$ which is represented by terms as above (they correspond natural numbers).
- **non-standard** natural numbers:
  They are natural numbers which does not have any corresponding digits, e.g. "infinite" natural numbers ∞.

Theory **T** is ω**-inconsistent** if any of its model contains non-standard natural numbers. Therefore, **FS**, $\mathbf{PAŁTr}_2$ is ω-inconsistent means that any of its model contains non-standard natural numbers.

There are some objections for ω-inconsistent truth theories.

- Vann McGee [M85] rejected the truth theory Γ within classical logic because of its ω-inconsistency.
- Hartry Field [Fl08] concluded that $\mathbf{PAŁTr}_2$ is not enough conservative because of ω-inconsistency.

3.3 Yablo ' s paradox

4 Analysis: Induction, coinduction and harmony

We have three paradoxes which causes the ω-inconsistency. The paradoxes suggest that, **many truth theories**, with full (or nearly full) T-scheme, **have a similar mechanism** which enables such derivations of paradoxes.

4.1 Interpreting γ coinductively

How can we interpret paradoxical sentences ? The provability of the consistency of **PA** and ω-inconsistency shares the same reason:

- **Tr** identify (real) formulae with Godel codes.
  - **Tr** interprets arithmetical operations on Godel codes to real operations on formulae.
  - Because of the existence of full T-scheme, we can identify Godel codes as formulae themselves.
  - So we can apparently define **recursive, infinitely operation on formulae**.
  - Recursive operations expands the definability of formulae. Truth predicate expands definability of operations of formulae:
- This makes us to define the following formulae: whose intuitive meaning looks infinite

  the modest liar infinite sentence (nested)

  $$A \equiv \neg A \to (\neg A \to (\neg A \to (\neg A \to \cdots)))$$

  McGee $\neg\mathbf{Tr}(\ulcorner\mathbf{Tr}(\ulcorner\mathbf{Tr}(\ulcorner\mathbf{Tr}(\ulcorner\cdots\urcorner)\urcorner)\urcorner)\urcorner)$

  (Non-standard length, e.g. infinite length!)

  Yablo $S_0 \equiv \neg\mathbf{Tr}(S_1)\bar{\wedge}(\neg\mathbf{Tr}(S_2)\bar{\wedge}(\neg\mathbf{Tr}(S_3)\bar{\wedge}\cdots))$ (we note that $S_i \equiv \neg\mathbf{Tr}(\ulcorner S_{i+1}\urcorner)\bar{\wedge}S_{i+2}$)
- The original sentences are finite, but they generate (or unfold) above infinite sentences (in that sense they are "**potentially infinite**").
- keyword: coinduction

Coinduction is a typical way to handle objects which is **potentially infinite** (the step to arrive at the initial case of the construction is delayed forever; as "free beer tomorrow").

4.1.1 (1) Recursive operations on formulae

- Truth predicate expands definability of operations of fromulae. : we can represent an infinite operation on a formula by some formula. are representable by arithmetization and **Tr**:
  - Remember: $0 \uparrow A \equiv A$ and $(n+1) \uparrow A \equiv \neg A \to (n \uparrow A)$.
  - In $\mathbf{PA\L Tr}_2$, formally speaking, $\uparrow$ is defined (as arithmetical function) as follows:
    (a) $f(0, x) = x$,
    (b) $f(n+1, x) = \dot{\neg} x \dot{\to} f(n, x)$.

(c) $n \uparrow A$ is just an abbreviation of $\mathbf{Tr}(f(n, \ulcorner A \urcorner))$.

- By using $(\exists y)[y \uparrow A]$, we can represent an infinite operation of taking a sup of $0 \uparrow A, 1 \uparrow A, \cdots$.

- **Tr** interprets **arithmetical operations on Godel codes** to **operations on formulae**.
  - Because of the existence of full **T**-scheme, we can identify Godel codes as formulae themselves.
  - So we can apparently define recursive operation on formulae.
- Recursive operations guarantees productivity.

Actually we can show the following:

- Let $\langle \varphi_i : i \in \omega \rangle$ be a sequence of formulae such that it is a recursive enumeration, i.e. there is a recursive function g such that $g(i) = \ulcorner \varphi_i \urcorner$.
- Then,
  - the co-inductive formulae $\varphi_0 \wedge (\varphi_1 \wedge (\cdots))$ is represented by $\forall x \mathbf{Tr}(g(x))$,
  - $\varphi_0 \vee (\varphi_1 \vee (\cdots))$ is represented by $\exists x \mathbf{Tr}(g(x))$ whose Godel code is a member of **Form**.

4.1.2 (2) Fixed point construction

- A simple example:
  - Let $f(n)$ be a recursive function which represents "self-conjunction operation", i.e.

$$f(\ulcorner A \urcorner) = \ulcorner A \,\bar{\wedge}\, \mathbf{Tr}(\ulcorner A \urcorner) \urcorner$$

  - Godel's fixed point lemma shows that there is a sentence λ s.t.

$$\lambda \equiv \mathbf{Tr}(f(\ulcorner \lambda \urcorner))$$

  - This λ satisfies (if $\ulcorner \lambda \urcorner$ is a standard natural number)

$$\lambda \equiv \mathbf{Tr}(\ulcorner \lambda \,\bar{\wedge}\, \mathbf{Tr}(\ulcorner \lambda \urcorner) \urcorner)$$

    (Nested box!)

- In this way, fixed point lemma together with recursive operations allows **co-inductive definition** of formulae (with some exception on non-standardness: see the next section).

4.1.3 paradoxes

- **慎ましい嘘つき**
  - Productivity: $f_0(n+1, \ulcorner A \urcorner) = \ulcorner \neg A \to f_0(n, \ulcorner A \urcorner) \urcorner)$,
  - Fixed point: $\lambda \equiv (\forall x)\mathbf{Tr}(f_0(x, \lambda))$,
  - Intuitive meaning: $\lambda \equiv \neg\lambda \to (\neg\lambda \to (\cdots))$

- **マクギー**
  - Productivity: $f_1(n+1, \ulcorner A \urcorner) = \ulcorner \mathbf{Tr}(f_1(n, \ulcorner A \urcorner)) \urcorner)$,
  - Fixed point: $\lambda \equiv \neg(\forall x)\mathbf{Tr}(f_1(x, \ulcorner \lambda \urcorner))$,
  - Intuitive meaning: $\lambda \equiv \neg\mathbf{Tr}(\ulcorner \mathbf{Tr}(\cdots) \urcorner)$
- **ヤブロー**
  - Productivity: $f_2(n+1, \ulcorner A(x) \urcorner) = \ulcorner \neg\mathbf{Tr}(\ulcorner A(n+1) \urcorner)\bar{\wedge}\mathbf{Tr}(f_2(n, \ulcorner A(n) \urcorner)) \urcorner$,
  - Fixed point: $\lambda(0)$ such that $\lambda(y) \equiv (\forall x)\mathbf{Tr}(f_2(x, \ulcorner \lambda(y) \urcorner)$,
  - Intuitive meaning: $\lambda(0) \equiv \neg\lambda(1)\bar{\wedge}(\neg\lambda(2)\bar{\wedge}(\neg\lambda(3)\bar{\wedge}\cdots))$
    - $S_i \equiv \lambda$ for any i,
    - Therefore $S_0 \equiv \neg S_1\bar{\wedge}\neg S_2\bar{\wedge}\cdots$ holds.

4.2 PTS for inductive formulae

To compare with the coinduction, first let us introduce a typical example of the **inductive definition.**

- **For any set A, the list of$A$ can be constructed as $\langle A^{<\omega}, \eta : (1+(A \times A^{<\omega})) \to A^{<\omega} \rangle$ by:**
  - **the first step: empty sequence $\langle\rangle$**
  - **the successor step: For all $a_0 \in A$ and sequence $\langle a_1, \cdots, a_n \rangle \in A^{<\omega}$**

$$\eta(a_0, \langle a_1, \cdots, a_n \rangle) = \langle a_0, a_1, \cdots, a_n \rangle \in A^{<\omega}$$

  The inductive definition corresponds to the existence of the least fixed point of η.

4.2.1 Thinking its meaning

Gentzen (1934, p. 80)

> **The introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions.**

The meaning of induction is given by the introduction rule,

- **the constructor η represents the introduction rule For any $a_0 \in A$ and sequence $\langle a_1, \cdots, a_n \rangle \in A^{<\omega}$,**

$$\eta(a_0, \langle a_1, \cdots, a_n \rangle) = \langle a_0, a_1, \cdots, a_n \rangle \in A^{<\omega}$$

- **the elimination rule γ is determined in relation to the introduction rule which satisfies the HARMONY**

4.2.2 The elimination rule and recursive computation

Prawitz (1965, p. 33) “inversion principle”

> **Let α be an application of an elimination rule that has β as consequence. Then, deductions that satisfy the sufficient condition [...] for deriving the major premiss of α, when combined with deductions of the minor premisses of α (if any), already “contain” a deduction of β; the deduction of β is thus obtainable directly from the given deductions without the addition of α.**

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ A\end{matrix}\quad\begin{matrix}\vdots\\ \beta\end{matrix}}{A\wedge\beta}\wedge+\qquad\begin{matrix}[v:A]\\ \vdots\\ \beta\end{matrix}}{\beta}\,\alpha \qquad\Rightarrow\qquad \begin{matrix}\vdots\\ A\\ \vdots\\ \beta\end{matrix}$$

The proof by the elimination rule should have the corresponding recursive function:

- **Input: the proof tree of β with α lefthand-side**
- **Output: the proof tree of β without α righthand-side**

Thinking its meaning (revised)

- **the constructor η represents the introduction rule For any $a_0 \in A$ and sequence $\langle a_1, \cdots, a_n\rangle \in A^{<\omega}$,**

$$\eta(a_0, \langle a_1, \cdots, a_n\rangle) = \langle a_0, a_1, \cdots, a_n\rangle \in A^{<\omega}$$

- **the elimination rule γ is represented as a recursive function over the datatype of formulae inductively defined by the introduction rule.**

The recursive function can be identified with the recursive procedure of the cut elimination or the normalization of proofs.

4.3 PTS for coinductive formulae

- **infinite streams of A is defined as $\langle A^\infty, \gamma : A^\infty \to (A \times A^\infty)\rangle$:**
  - **Their intuitive meaning is infinite streams of the form $\langle a_0, a_1, \cdots\rangle \in A^\infty$**

  $$\gamma(\langle a_0, a_1, \cdots\rangle) = (a_0, \langle a_1, \cdots\rangle) \in (A \times A^\infty)$$

  γ is a function which pick up the first element a_0 of the stream. (but $\langle a_1, \cdots\rangle$ is still infinite stream) for $\langle a_0, a_1, \cdots\rangle \in A^\infty$.

- **Coinduction represents the intuition such that finitely operations on the first element of the infinite sequence is possible to compute(Productivity).**

Actually NEC, CONEC only care about the productivity (or 1-step computation)!

$$\frac{\varphi}{\mathbf{Tr}(\varphi)}\ \mathbf{NEC} \qquad \frac{\mathbf{Tr}(\varphi)}{\varphi}\ \mathbf{CONEC}$$

4.3.1 Thinking its meaning

The coinductive definition is a dual of the inductive definition. Therefore all roles are opposite. The meaning of coinduction is by the elimination rule

- **de-constructor γ represents the elimination rule [S12], for any infinite stream $\langle a_0, a_1, \cdots \rangle \in A^\infty$,**

$$\gamma(\langle a_0, a_1, \cdots \rangle) = (a_0, \langle a_1, \cdots \rangle) \in (A \times A^\infty)$$

  γ is a function which picks up the first element a_0.

- **the introduction rule is known to have to satisfy the condition (the guarded corecursion) which corresponds to the failure of the formal commutativity of Tr.**

The final algebra which has the same structure to streams is called weakly final coalgebra

What is a guarded corecursive function? Remember: a property of stream = infinite as a whole, but 1 step computation is possible. Guarded corecursion guarantees the productivity of functions over coinductive datatypes.

- **for any recursive function f, map : $(A \to B) \to A^\infty \to B^\infty$ is defined as**

$$\mathbf{map}\, f\, \langle x, x_0, \cdots \rangle = \underbrace{\langle f(x) \rangle}_{\text{finite part}} \frown \underbrace{(\sharp\, \mathbf{map}\, f \langle x_0, x_1, \cdots \rangle)}_{\text{infinite part}}$$

  where $\frown$ means the concatenation of two sequence.

- **Here recursive call of map only appears inside $\sharp$, the inside and outside of $\sharp$ are essentially different as if the those of modal operators are different, e.g. $\Box(\varphi \wedge \psi)$ and $(\Box\varphi) \wedge \psi$ are different.**

The introduction rule and guarded corecursive function

Analogy to the inductive case: the proof by the introduction rule should have the corresponding guarded corecursive function:

- **Input: the (coinductive) proof tree with the use of the introduction rule**
- **Output: the proof tree without the use of the introduction rule**

Actually T-schemata only care about the productivity (or 1-step computation)! the elimination rule excludes the first Tr in the formula!

$$\frac{\varphi}{\mathbf{Tr}(\varphi)}\ \mathbf{NEC} \qquad \frac{\mathbf{Tr}(\varphi)}{\varphi}\ \mathbf{CONEC}$$

the introduction rule should correspond that; it should represent 1 step computability
= guarded corecursion

4.4 Non-terminate computations

What is a cause of problem?

- **Problem:** Formulae of $\mathbf{PAŁTr}_2$ are not isomorphic to **Form**, a set of Godel codes of formulae in $\mathbf{PAŁTr}_2$.
- **Remember:**
  - formulae of $\mathbf{PAŁTr}_2$ are inductively defined in the **meta-theory**,
  - **Form** contains co-inductive formulae defined in the **object theory**.
- **Known:** It is said that **unrestricted form of coinduction is contradictory to induction**.
  - Example: Martin-Lof's Intuitionistic Type Theory case:
  - By using coinduction, we can define **functions whose computation does not terminate** in finite many steps.
  - Matin-Lof's ITT's principle, which is based on induction, is that "computations of any definable functions must terminates eventually: contradiction!
- In this sense, the failure of **FC** is due to the **conflict** between
  - the inductive definition in the **meta level**
  - and the co-inductive definition in the **object level**.
- **Remember:** there is **no** conflict between
  - the induction principle in the **object level**
  - and the co-inductive definition in the **object level**.

  That is due to the following facts:

  - Godel codes of potentially infinite objects are related to the non-standard natural numbers,
  - such non-standard numbers are infinite object from the viewpoint of the meta theory,
  - but they are finite object from the viewpoint of the object theory.
- In the case of **FC**, there is no way-out because the inductive definition is done in the meta-level.

From the viewpoint of the meta level

- Formulae, which are defined co-inductively in the object theory, are of the form $\mathbf{Tr}(\cdots)$,
- if we think of their intuitive meaning, they are infinite objects in the sense of the meta theory,
- however the inductive definition of formulae in the meta theory requires they should be finite object in the sense of the meta theory,
- therefore the correspondence between co-inductively defined formulae and formulae are not perfect.

5 Solution: The HARMONY extended

Non-normalizability of the proof; subsection?

5.1 The failure of FC

5.2 The guarded corecursion and the failure of Formal commutability

Remember:

ω-inconsistency = **PA** + the **formal commutability** of **Tr** +**NEC**+**CONEC**

- The **formal commutability** violates the **guardedness**!

$$\frac{\mathbf{Tr}(\ulcorner\gamma \to \neg\mathbf{Tr}(\ulcorner\mathbf{Tr}(\ulcorner\cdots\urcorner)\urcorner)\urcorner)}{\mathbf{Tr}(\ulcorner\gamma\urcorner) \to \neg\mathbf{Tr}(\ulcorner\mathbf{Tr}(\ulcorner\mathbf{Tr}(\ulcorner\cdots\urcorner)\urcorner)\urcorner)}\ FS$$

Both γ and $\mathbf{Tr}(\ulcorner\mathbf{Tr}(\ulcorner\mathbf{Tr}(\ulcorner\cdots\urcorner)\urcorner)\urcorner)$ are coinductive objects:

  - To calculate the value of $\mathbf{Tr}(\ulcorner\gamma \to \neg\mathbf{Tr}(\ulcorner\mathbf{Tr}(\ulcorner\cdots\urcorner)\urcorner)\urcorner)$, we should calculate the both the value of $\mathbf{Tr}(\ulcorner\gamma\urcorner)$ and $\neg\mathbf{Tr}(\ulcorner\mathbf{Tr}(\ulcorner\mathbf{Tr}(\ulcorner\cdots\urcorner)\urcorner)\urcorner)$.
  - But $\mathbf{Tr}(\ulcorner\gamma\urcorner)$ is **also coinductive object**, therefore the calculation of the first value $\mathbf{Tr}(\ulcorner\gamma\urcorner)$ **never terminates**,
  - therefore this violates the productivity.

- Therefore **Tr** with the **formal commutability** cannot be a logical connective!!

The solution

If we abandon the **formal commutability**, we can regard **Tr** as a logical connective in the extended sense.

Let us call $\mathbf{FS}^* = \mathbf{PA} + \mathbf{NEC} + \mathbf{CONEC}$;
$\mathbf{FS}^*$ proves the following

- We can prove the commutativity for any concrete "real" formula as follows: (In this sense, the failure of the formalized commutativity is not a serious problem on truth conceptions)
  - for any atomic formula ψ, $\mathbf{Tr}(\ulcorner\psi\urcorner) \equiv \psi$,
  - for any formula φ, $\mathbf{Tr}(\ulcorner\neg\varphi\urcorner) \equiv \neg\mathbf{Tr}(\ulcorner\varphi\urcorner)$,
  - for any formulae φ, ψ, $\mathbf{Tr}(\ulcorner\varphi \wedge \psi\urcorner) \equiv \mathbf{Tr}(\ulcorner\varphi\urcorner) \wedge \mathbf{Tr}(\ulcorner\psi\urcorner)$, $\mathbf{Tr}(\ulcorner\varphi \vee \psi\urcorner) \equiv \mathbf{Tr}(\ulcorner\varphi \vee \mathbf{Tr}(\ulcorner\psi\urcorner)$, $\mathbf{Tr}(\ulcorner\varphi \to \psi\urcorner) \equiv \mathbf{Tr}(\ulcorner\varphi\urcorner) \to \mathbf{Tr}(\ulcorner\psi\urcorner)$,
  - for any formula φ, $\mathbf{Tr}(\ulcorner\forall z\, \varphi(z)\urcorner) \equiv \forall z\mathbf{Tr}(\ulcorner\varphi(z)\urcorner)$, $\mathbf{Tr}(\ulcorner\exists z\, \varphi(z)\urcorner) \equiv \exists z\mathbf{Tr}(\ulcorner\varphi(z)\urcorner)$.
- The theory might be ω-consistent.

5.2.1 Model construction of $\mathbf{FS}^*$ + ω-consistency

5.3 Truth predicate as a de-constructor of a coinductive datatype of formulae

In $\mathbf{FS}^*$, we can say the following:

- the meaning of truth predicate is given by the elimination rule (**CONEC**)

- the introduction rule is given by the guarded corecursion, this corresponds to the failure of the formalized commutation scheme (FC).

  If we extend the concept of HARMONY for coinductive objects, then we can say "**Tr** in **FS*** is a logical connective".

  - **The nature of truth predicate here is to make it possible to define coinductive formula of infinite length.** The truth predicate is a deconstructor of such coinductive type.
  - The truth conception here guarantees the **reducibility** of the argument,
  - the essence is that the truth predicate never prevent the reducibility from the assumption to the consequence.

6 Conclusion

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