ON EVANS'S VAGUE OBJECT FROM SET THEORETIC VIEWPOINT

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Abstract. Gareth Evans proved that if two objects are indeterminately equal then they are different in reality. He insisted that this contradicts the assumption that there can be vague objects. However we show the consistency between Evans's proof and the existence of vague objects within classical logic. We formalize Evans's proof in a set theory without the axiom of extensionality, and we define a set to be vague if it violates extensionality with respect to some other set. There exist models of set theory where the axiom of extensionality does not hold, so this shows that there can be vague objects.

§1. Introduction. In his short paper "Can There Be Vague Objects?" [5], Gareth Evans questioned a consistency between two assumptions: Vagueness is 'a necessary feature of any true description' of the world, and 'amongst the statements which may not have a determinate truth value as a result of their vagueness are identity statements'. He defined vague objects as having vague identity statement: a is a vague object if there exists an object b such that $a = b$ is of indeterminate truth value. Let us assume *there can be vague objects in the world*; we call this Evans's Vagueness Assumption (**EVA**). He proceeded with his argument as follows: Let a, b be vague objects, then

(I)	$\bigtriangledown(a = b)$,	i.e. $a = b$ is indeterminate (assumption),
(II)	$\lambda x[\bigtriangledown(a = x)]_b$,	i.e. b is indeterminately equal to a (from (I)),
(III)	$\neg \bigtriangledown (a = a)$,	i.e. $a = a$ is determinate,
(IV)	$\neg\lambda x[\bigtriangledown(a = x)]_a$,	i.e. a is not indeterminately equal to a (from (III)),
(V)	$a \neq b$,	i.e. a is not equal to b (from (II) and (IV)).

We note that $\bigtriangledown\varphi$ means that the truth value of φ is indeterminate. The conclusion (V) is 'contradicting the assumption that the identity statement "$a = b$" is of indeterminate truth value'. Hence (I) must be rejected, that is to say, any identity statement has determinate truth value. Therefore, he seems to conclude **EVA** does not hold.

Key words and phrases. axiom of extensionality, classical logic, extensionality, set theory, vagueness, vague object.

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Some philosophers have agreed with his conclusion[1]. For example, Brian Garrett denied the possibility of vague identity and vague object [7]. However, many articles have been published against Evans's conclusion. One typical approach is to analyze his proof within many-valued logic. For example, Jack Copeland tried to prove that the derivation of (V) from (I) is not valid within fuzzy logic without mentioning **EVA** [3]. Michael Tye accepted **EVA** and represented vagueness within many-valued logic: membership relation $x \in y$ can be a vague predicate, i.e. its truth value is neither 0 nor 1 for some sets x and y [18]. However, it has been objected that 'the writers who adopt this strategy rarely provide much argument for the need for a many-valued logic' [11, 55]. Another approach takes the modal point of view. For example, Ken Akiba defined a vague object as a transworld object. He distinguished identity relation from coincidence relation on identity in Evans's proof, and then tried to show that Evans's proof holds true only for the case of identity statement [1].

In this paper, we defend both **EVA** and Evans's derivation from (I) to (V): We show that **EVA** does not imply a contradiction even within classical logic and nevertheless Evans's derivation from (I) to (V) is correct. In many-valued logic and modal approaches, some non-classical logic is required to represent vague object. However, we need not to require any non-classical logic: We can regard Evans's proof as being done within classical logic[2], and it is not necessary to abandon classical logic to represent a vague object.

We claim that, among other properties, extension is worth being focused on when we consider a vague object. In fact, philosophical discussions about vagueness often begin with explaining or sometimes defining it in terms of extension [11]. Now, one of the simplest frameworks to consider extension is set theory, so we employ set theory in this paper. The key to formalize Evans's proof in set theory is to interpret his word "*indeterminate*". There are many ways to interpret it. For example, it is interpreted as "its truth value is neither 0 nor 1" in many-valued logic, or it is represented by using a modal operator in modal logic. However, we regard the truth value of any formula as determinate, and we add neither a new predicate nor operator which represent indeterminacy. We can interpret '$a = b$ is indeterminate' as some set-theoretic property, namely *the axiom of extensionality is violated for a and b*, which is definable by membership relation. We will demonstrate that a formalization of Evans's proof in set theory justifies the interpretation above.

Technically speaking, Evans's proof seems to have three implicit assumptions as follows:

(i) For every a, $a = a$ has definite truth value $(\neg \triangledown (a = a))$,
(ii) *the Diversity of the Dissimilar* (**DD**) : if object a has a property that b lacks, then you can infer $a \neq b$,

[1] In this paper 'vague object' is meant in Evans's sense. Some philosophers insisted that Evans's proof merely shows that vague object in his sense does not exist. They say, there could be other kinds of vague object and, as Harold Noonan wrote, 'it is consistent to hold both that there are vague objects and that the identity relation is precise' [16]. However, we do not consider these kinds of vague objects because we concentrate on Evans's proof.

[2] Even if we regard Evans's derivation as being done within modal logic, modality does not play an essential role in the derivation of (V) from (I): **DD** eliminates all operators in (V).

(iii) $\vdash \varphi$ implies $\vdash \triangle\varphi$ (as the generalization law in $\mathbf{S_5}$-modal logic).

For more details, see [11]. We note that $\triangle\varphi$ means that the truth value of φ is determinate. (ii) is used to infer (V) from (II) and (IV). Now, $\triangle(a \neq b)$ is inferred from (V) and (iii). For duality[3], $\neg \bigtriangledown (a = b)$ is inferred from $\triangle(a \neq b)$. This contradicts (I). But, we can disregard (iii). Indeed, (i) and (ii) are necessary to derive (V) from (I), but (iii) has nothing to do with the derivation itself. Then, if we do not admit (iii), what Evans proved is merely that *the vague identity statement (I) implies (V).* We call '$\bigtriangledown(a = b) \to a \neq b$' Evans Conditional (**EC**) as in [3]. We show that our definition of indeterminate equality satisfies **EC**, and that both **EVA** and **EC** are consistent when we employ a set theory without the axiom of extensionality.

§2. Formalizing Evans's proof in set theory. In this section, we attempt to formalize Evans's proof in set theory. We seek a way of formalizing axioms of set theory with operators $\triangle, \bigtriangledown$ which satisfies (i) and (ii) but not (iii). The difficulty lies in formalizing inferences which have as conclusion formulas of the form $\triangle(a \neq b)$ or $\bigtriangledown(a = b)$. As a basis for discussion, we notice such an inference rule proposed by Harold Noonan. However, it is known that his rule implies a contradiction if it is applied to Evans's proof. So, we must examine his rule and restrict it not to imply a contradiction. At the end of this section, we formalize Evans's proof in set theory and show it is valid (for a complete list of axioms and their consistency, see Appendix).

In Evans's proof, two kinds of relation are used: Leibniz equality relation and vague equality relation. The confusion of these relations seems to make Evans's proof paradoxical, so it is important to distinguish them. The famous relations are as follows:

Leibniz equality: $x = y$ iff $(\forall z)[(z \in x \leftrightarrow z \in y) \,\&\, (x \in z \leftrightarrow y \in z)]$,
Extensional equality: $x =_{\text{ext}} y$ iff $(\forall z)[z \in x \leftrightarrow z \in y]$.

Of course $x = y \to x =_{\text{ext}} y$ holds. Since $(\forall z)[x \in z \leftrightarrow y \in z] \to (\forall z)[z \in x \leftrightarrow z \in y]$ holds[4], the definition of Leibniz equality is usually written as $x = y$ iff $(\forall z)[x \in z \leftrightarrow y \in z]$. The Leibniz law $a = b \to (\varphi(a) \leftrightarrow \varphi(b))$ surely holds for Leibniz equality, however there is no guarantee that it holds for extensional equality. It is necessary to consider the axiom of extensionality when we think about identity relations: The axiom of extensionality guarantees that, for any set x and y, $x =_{\text{ext}} y \to x = y$.

Our purpose here is to analyze equality relation separately from membership relation, so we decide to require the following axiom in a single uniform way: For any set x, y,

$$x \in y \to \neg \bigtriangledown (x \in y) \tag{1}$$

[3]Duality between $\triangle$ and $\bigtriangledown$ is valid: $\neg \triangle \neg\varphi \leftrightarrow \bigtriangledown\varphi$.

[4]By definition, when $a \neq_{\text{ext}} b$, there is a set c such that $c \in a \,\&\, c \notin b$ or vice versa. So the axiom of separation and the axiom of power set guarantee that, a set D such that $(\forall x)[x \in D \leftrightarrow (\forall y)[y \in x \to y \in a] \,\&\, c \in x]$ exists, and it distinguishes a and b; $a \in D$ and $b \notin D$ holds. We remind that such a way of distinction is the same as Evans's proof.

In our framework, vagueness appears not in the truth value of membership relation as fuzzy set theory or Tye's set theory[5], but in another property which concerns the equality relation. We also require duality of $\bigtriangledown$ and $\triangle$.

As for **DD**, by interpreting λ-term in Evans's proof[6], we formalize *the Diversity of the Dissimilar for Sets*

$$\mathbf{DDS}: \ (\forall x, y)(\exists X)[x \in X \,\&\, y \notin X \to x \neq y]$$

It is clearly true from definition of Leibniz equality.

DDS never tells us anything about $\triangle, \bigtriangledown$, so we need another (or a stronger) principle to examine their properties. However, there are few such principles; here we attempt to reformalize the most famous one, Noonan's formulation of Evans's proof [15]. He used *the Diversity of the Definitely Dissimilar*[7]

$$\mathbf{DDD}: \ \text{ from } (\triangle\varphi(a) \,\&\, \triangle \neg\varphi(b)), \text{ you can infer } \triangle\,(a \neq b)$$

As we work in set theory, we formalize this principle as the axiom scheme: *the axiom scheme of the Diversity of the Definitely Dissimilar* is,

$$(\triangle\varphi(a) \,\&\, \triangle \neg\varphi(b)) \to \triangle(a \neq b) \ \text{ for any formula } \varphi$$

Combining this scheme with the axiom (1), we formalize *the axiom of the Diversity of the Definitely Dissimilar for Sets*[8] as follows:

$$\mathbf{DDDS}: \ (\forall x, y)(\exists X)[x \in X \,\&\, y \notin X \to \triangle(x \neq y)]$$

DDDS says that X distinguishes x and y. Noonan showed that **DDD** implies the same conclusion as Evans that $\bigtriangledown(a = b)$ implies a contradiction, without using (iii). In this sense, admitting **DDD** is more than admitting (iii) and **DDDS** is too strong to be consistent.

Now, we remark about the axiom of extensionality. For any sets a and b, assume they are not extensionally equal. Fix any set x such that $x \in a$ but $x \notin b$ (or $x \in b$ but $x \notin a$ if it is not the case). Then $a \in X$ and $b \notin X$ where X is any set such that $(\forall y)[y \in X \leftrightarrow y \in Y \,\&\, x \in y]$ (guaranteed by the separation schema) and Y is any set such that $(\forall x)[x \in Y \leftrightarrow x = a \vee x = b]$ (guaranteed by the axiom of pairing)[9], so **DDDS** proves $\neg \bigtriangledown (a = b)$, i.e. $(\exists x)[x \in a \,\&\, x \notin b] \to \triangle(a \neq b)$. Taking its contraposition and combining it with **EC**, we can conclude the following:

$$\bigtriangledown(a = b) \to a =_{\text{ext}} b \,\&\, a \neq b \tag{2}$$

[5]Tye's set theory is an attempt to represent vagueness within Kleene's 3-valued logic. We insist that it is not enough to represent vagueness as it appears in Evans's proof. In Tye's theory, the truth value of $a = a$ is indefinite: This contradicts assumption (i). He wrote that it is a *quasi-tautology*; it must be true in any sharpened world, so we can regard it as almost-determinate statement (similarly he insisted that the axiom of extensionality is also a quasi-tautology). However, it is difficult to interpret "quasi-tautology" as "determinately true": He seems to justify both $a = a$ and the axiom of extensionality by supposing that they should be true.

[6]We interpret $\lambda y[P(y)]_x$ as $(\exists X)x \in Y_X$ where $(\forall y)[y \in Y_X \leftrightarrow y \in X \,\&\, P(y)]$.

[7]Here we employ its derivative version which is called 'DD$_3$' in [11, 54].

[8]We follow Noonan's notation but Evans's hypothesis (i) suggests to write $\neg\bigtriangledown x \neq y$ instead of $\triangle x \neq y$.

[9]We usually write $X = \{y : y \in \{a, b\} \,\&\, x \in y\}$ and $Y = \{a, b\}$ in **ZF**. However, we will work in the non-extensional set theory, and we cannot guarantee the uniqueness of X, Y there.

It shows that, whenever $a = b$ is indefinite, the axiom of extensionality is violated.

Of course, the violation of the axiom of extensionality itself does not imply a contradiction. The same is true of **EC**. However, **DDDS** implies a contradiction from (2). Let us assume $\triangledown(a = b)$. Fix any set A such that $(\forall x)[x \in A \leftrightarrow x = a]$ holds (guaranteed by the axiom of pairing), then **DDDS** implies $\triangle(a \neq b)$ because $a \in A$ and $b \notin A$. It contradicts the assumption $\triangledown(a = b)$.

To analyze this derivation in depth, we write it without using **DDDS**: Suppose $\triangledown(a = b)$ (we note that this means a and b are vague objects), then[10]

(a) $a \in A$ and $b \notin A$ (from **EC**: $a \neq b$)
(b) $\triangle(a \notin A)$ and $\triangle(b \in A)$ (from (a), the axiom (1) and duality of $\triangle$)
(c) $\triangle(a \neq b)$ (from (b) and the axiom scheme of **DDD**)

The point of this argument lies in (b), $\triangle(b \in A)$. It is clear that $b \in A \leftrightarrow a = b$ holds, and the axiom (1) insists $\triangle(b \in A)$ on the left-hand side even though $\triangledown(a = b)$ holds on the right-hand side. Such difference itself is not a contradiction, because $\triangle(b \in A)$ is a statement about membership but $\triangledown(a = b)$ is about equality relation. However, the axiom scheme of **DDD** connects both kinds of statements: $\triangle(b \in A)$ implies $\triangle(a \neq b)$, which contradicts the assumption. In this sense, the axiom (1) and **DDDS** (or the axiom scheme of **DDD**) are incompatible when the set A, which distinguishes a and b, contains a vague object.

As we saw, **DDS** is consistent nevertheless **DDDS** is too strong to be consistent when we assume the axiom (1). So the limit of consistency is between them. We will approximate where the limit is: Our strategy is that we restrict **DDDS** to an axiom which is weak enough not to imply a contradiction and strong enough to tell us something about $\triangle, \triangledown$[11].

But, how? It is observed that a contradiction is derived when A, which distinguishes two sets a and b in **DDDS**, contains a vague object a. As working assumption, we abandon this way of distinguishing *in the case that the set, which distinguishes two objects, contains some vague object*. And the weakest candidate for an alternative principle is to abandon such way of distinguishing, not only in the case when A contains a vague object, but also in the case when, for any set X, X distinguishes two objects x and y externally, i.e. $x \in X$ and $y \notin X$, in **DDDS**. We formalize *the Diversity of the Extensional Dissimilar*

$$\mathbf{DED}: \ (\forall x, y)(\exists z)[z \in x \,\&\, z \notin y \rightarrow \triangle(x \neq y)]$$

It represents $x \neq_{\mathrm{ext}} y \rightarrow \triangle(x \neq y)$; **DDDS** implies **DED** as we noticed above. We show **DED** is enough strong to prove some property of $\triangledown$.

[10] As for the derivation of (b) from (a), Noonan argued that $\lambda x[\triangledown(a = x)]_b$ in Evans's proof is definitely true so we can conclude that $\triangle \lambda x[\triangledown(a = x)]_b$. Similarly $\triangle \neg \lambda x[\triangledown(a = x)]_a$ holds. Therefore, we can conclude $\triangle a \neq b$. It was objected that allowing such derivations strengthens **DDD** in effect [10]. However, we do not take this up here because we investigate vagueness when any membership is definitive.

[11] Such strategy is known as effective way to find the limit empirically in set theory. Let us remember the solution of Russell paradox within classical logic. We restrict the comprehension principle to the axioms, **ZFC**, which is weak enough not to imply a contradiction, and strong enough to express classical mathematics.

Taking contraposition of **DED** and combining it with **EC**, we can conclude (2). So, if we employ **DED** principle, we can formalize Evans's proof in set theory as follows:

(I') $\triangledown(a = b)$,
(I") $a =_{ext} b \,\&\, a \neq b$ (from (2)),
(II') $b \in X$ where X is such that $(\forall x)[x \in X \leftrightarrow x \in Y \,\&\, a =_{ext} x \,\&\, a \neq x]$ and Y is such that $(\forall x)[x \in Y \leftrightarrow x = a \vee x = b]$ (from (I')),
(III') $\neg(a =_{ext} a \,\&\, a \neq a)$,
(IV') $a \notin X$ (from (III')),
(V') $a \neq b$ (from **DDS**)

This shows that **EC** is always valid in our set theory. It does not imply a contradiction (for the proof, see the next section and Appendix). This is because the theory does not have any principle which derives $\triangle(a = b)$ from (V').

§3. Models of EVA and EC. As we saw in the previous section, **DED** implies (2): We again take notice of the violation of the axiom of extensionality. The axiom of extensionality can be seen as a representation of precision since any set is determined precisely by its members. In this sense, the violation of the axiom of extensionality represents some aspect of vague object. This means that the converse of (2) holds in set theory. So we regard such violation of this axiom as a representation of vagueness here[12]:

$$\triangledown(a = b) \leftrightarrow a =_{\text{ext}} b \,\&\, a \neq b \tag{3}$$

As we saw, vague object is defined by using vague identity, i.e. a is a vague object if and only if $(\exists x) \triangledown (a = x)$. So we call a vague object when this axiom is violated. More precisely,

DEFINITION 1. *a is a vague object iff the axiom of extensionality is violated for a, i.e.*

$$(\exists x)[a =_{ext} x \,\&\, a \neq x]$$

Assuming Definition 1, **EVA** implies a contradiction only when we assume the axiom of extensionality. Otherwise, $\triangledown(a = b)$ implies $a \neq b$ without implying $\triangle(a \neq b)$. So (3) and Definition 1 show that any model of set theory in which the negation of the axiom of extensionality holds is a model of **EVA** and **EC**. There exist many such models, so this fact proves consistency of our definition.

We can easily generalize Definition 1: The violation of the axiom of extensionality represents vagueness not only within classical logic but also within a greater variety of logics. So, within any logic, we insist that set theory without the axiom of extensionality is required to represent vague object. Conversely,

[12] We remark that George Boolos [2] insisted that the axiom of extensionality is so essential that any collection of objects which violate this axiom is unqualified to be called *set*. However, we do not agree with him: His motivation seems to analyze **ZF**, however many other set theories without the axiom of extensionality have been studied and such theories seem to say something valid about some aspects of *set*.

many such set theories have been proposed by now[13]. These theories seem to give an example of vague object in the sense of Definition 1.

§4. Conclusion. In this paper, we examined Evans's proof from a set theoretic viewpoint. We defined vague objects as objects for which the axiom of extensionality does not hold within classical logic, so we could construct a model of **EVA** and **EC** (at the same time we reformulated **DD** to **DED**). This means that the assumption that *there can be vague objects in the world* itself does not imply a contradiction nevertheless Evans's proof is still valid. Namely, if you accept our definition of vague objects, you can conclude that there can be vague object in the world.

Needless to say, we must inquire further into vague object. We must investigate what properties does vagueness, represented by violation of the axiom of extensionality, have. It seems to have many interesting aspects: For example, it can be regarded as vagueness without boundary. R.M. Sainsbury wrote that 'a description of concepts or predicates in terms of what sets they determine is a description of them as boundary drawers' [17, 252]. However, we note that, when $a =_{\mathrm{ext}} b \,\&\, a \neq b$, a and b share the same boundary. In this sense, our vagueness

[13]Traditionally, this has been studied within intuitionistic logic; one of the most famous results is due to Harvey Friedman [6].

V.N.Grisǐn showed that the comprehension principle alone does not imply Russell paradox within Grisǐn logic, which is classical logic minus the contraction rule [8]. He also showed that the comprehension principle and the axiom of extensionality are incompatible within Grisǐn's logic: the axiom of extensionality implies the contraction rule (so this implies Russell paradox) in set theory with the comprehension principle within Grisǐn logic.

Peter Hajek and Zuzana Hanikova developed Fuzzy Set Theory **FST** [9], within the framework of fuzzy logic with operator $\triangle$ which means 'determinately true', i.e. the truth value of $\triangle\varphi$ is 1 if the value of φ is already 1; otherwise, the truth value of φ is less than 1, then $\triangle\varphi$ takes value 0 (in BL-chains). It is in the style of **ZF**, and it seems to be an attempt to axiomatize our intuition of fuzzy set. In **FST**, the axiom of extensionality cannot be valid. This is because that Leibniz equality becomes crisp (i.e. its truth value is 0 or 1) nevertheless the truth value of extensional equality can be indeterminate: They proved the following lemma.

LEMMA 1. *A theory with comprehension (for open formulas) or pairing (or singletons) over a logic which proves the propositional formula* $(\varphi \to \varphi \,\&\, \varphi) \to (\varphi \vee \neg\varphi)$ *proves* $(\forall x, y)[x = y \vee x \neq y]$*, i.e. Leibniz equality is crisp.*

For the proof, see [9, §4]. We note that fuzzy logic proves $(\varphi \to \varphi \,\&\, \varphi) \to (\varphi \vee \neg\varphi)$, and **FST** has the axiom of pairing. It is clear that the truth value of extensional equality could be any real value between 0 and 1. Here, the axiom of extensionality holds for any crisp set, but it might be violated for some fuzzy set: The truth value of Leibniz equality and that of extensional equality might be different for some fuzzy set. So **FST** can only have the weakened version of the axiom of extensionality: $x = y$ iff $\triangle(x \subseteq y) \& \triangle(y \subseteq x)$. Such violation of the axiom has been regarded as merely introduced for technical reasons, however this seems to suggest that such violation is a necessary feature of fuzzy set implicitly connoted by our intuition of fuzziness itself. In this sense, Definition 1 can be regarded as an isolation of some aspect of fuzziness so that we can represent it even within classical logic.

As for modal logic, Jan Krajicek developed the Modal Set Theory **MST** [13] [14]. It has an operator $\Box$ which represents 'to be knowable', and it is an axiomatization of a set theory based on a modal version of the comprehension axioms as the only non-logical axioms. Unfortunately the consistency of **MST** is still an open problem. It is worthy of special mention that **MST** disproves the axiom of extensionality: Therefore such the similarity, with Grisǐn's and with ours, is worthy of attention.

is an example of vagueness without boundary, nevertheless it is represented in terms of sets.

Another problem is studying the relationship between vague object and vague predicate. Vague object is an object for which identity relation is a vague predicate. Some philosopher, as Michael Dummett [4], claimed that vague predicate contains inconsistency, so, no semantics of vague predicate can be given. But so far, we can give a model of vague object in classical logic. So, if the relationship between vague object and vague predicate becomes clearer, the possibility of existence of semantics of vague predicate would remain.

Appendix: The axioms of set theory with operators $\bigtriangledown$, $\bigtriangleup$ and their consistency. For the reader's convenience, we give an analysis of our set theory with two operators $\bigtriangledown$, $\bigtriangleup$. Let T be any consistent set theory within classical logic with enough power of expression and with the negation of the axiom of extensionality.

The axioms. We expand T to T^* by adding the following axioms. We note that the language of T^* contains two predicates $\in$ and $=$, and two operators $\bigtriangleup, \bigtriangledown$.

Definition 2. *The added axioms and axiom schemata are as follows:*

(A) *The membership relation is definite, i.e.* $(\forall x, y) x \in y \to \neg \bigtriangledown x \in y$.
(B) $\bigtriangledown$ *and* $\bigtriangleup$ *are dual, i.e.* $\bigtriangledown \varphi \leftrightarrow \neg \bigtriangleup \neg \varphi$ *for any* φ.
(C) φ *is definite (indefinite) iff* $\neg\varphi$ *is definite (indefinite), i.e.* $\bigtriangledown\varphi \leftrightarrow \bigtriangledown\neg\varphi$ *and* $\bigtriangleup\varphi \leftrightarrow \bigtriangleup\neg\varphi$ *for any* φ.
(D) *Our definition (3), i.e.* $\bigtriangledown(a = b)$ *iff* $(a =_{ext} b \,\&\, a \neq b)$.

The interpretation. From now on, we write $a \overset{\bigtriangleup}{=} b$ instead of $a = b \vee a \neq_{ext} b$, and $a \overset{\bigtriangledown}{=} b$ instead of $a =_{ext} b \,\&\, a \neq b$ just for simplicity. To show the consistency of T^*, we provide an interpretation of sentences of T^* into T inductively as follows. Let us suppose that $\langle \varphi_i : i \in \omega \rangle$ is an enumeration of all formulae such that

- if φ_i is a subformula of φ_j then $i \leq j$ holds,
- if φ_k is of the form $\bigtriangleup\varphi_l$ (or $\bigtriangledown\varphi_l$) then $l \leq k$ holds,
- if φ_m is of the form $(Qx)\varphi_n(x)$ then $n \leq m$ holds where Q is either $\forall$ or $\exists$.

We note that the list $\langle \varphi_i : i \in \omega \rangle$ guarantees the uniqueness of the order of interpretation.

Definition 3. *For any formula* $\varphi \equiv \varphi_i$, *we define the following interpretation inductively along the list* $\langle \varphi_i : i \in \omega \rangle$*:*

(1) (1a) *if there is some* $j < i$ *such that* φ_i *is equivalent to* φ_j *then* $\bigtriangledown\varphi_i$ $(\bigtriangleup\varphi_i)$ *is interpreted as the same formula as* $\bigtriangledown\varphi_j$ $(\bigtriangleup\varphi_i)$*,*
(1b) *if there is some* $j < i$ *such that* φ_i *is equivalent to* $\neg\varphi_j$ *then* $\bigtriangledown\varphi_i$ $(\bigtriangleup\varphi_i)$ *is interpreted as the same formula as* $\bigtriangledown\varphi_j$ $(\bigtriangleup\varphi_i)$*,*
(2) φ *is interpreted as* φ *if it includes neither* $\bigtriangledown$ *nor* $\bigtriangleup$ *(and if it is not the case (1)),*
(3) *if* $\varphi(x_0, \cdots, x_{n-1})$ *has only* n *free variables,*

(3a) $\triangledown\varphi(a_0,\cdots,a_{n-1})$ *is interpreted as*

$$\neg\varphi(a_0,\cdots,a_{n-1})\,\&\,[(\exists x_0)(x_0 \overset{\triangledown}{=} a_0\,\&\,\varphi(x_0,a_1,\cdots,a_{n-1})) \vee\cdots\vee(\exists x_{n-1})(x_{n-1}\overset{\triangledown}{=} a_{n-1}\,\&\,\varphi(a_0,\cdots,a_{n-2},x_{n-1}))]$$

(3b) $\triangle\varphi(a_0,\cdots,a_{n-1})$ *is interpreted as*

$$\varphi(a_0,\cdots,a_{n-1})\vee[(\forall x_0)(a_0\overset{\triangle}{=} x_0\vee\neg\varphi(x_0,a_1,\cdots,x_{n-1})) \,\&\,\cdots\,\&\,(\forall x_{n-1})(a_{n-1}\overset{\triangle}{=} x_{n-1}\vee\neg\varphi(a_0,\cdots,a_{n-2},x_{n-1}))]$$

(3c) $\triangledown(Qx_0,\cdots,Qx_{n-1})\varphi(x_0,\cdots,x_{n-1})$ *is interpreted as* $(Qx_0,\cdots,Qx_{n-1})\triangledown\varphi(x_0,\cdots,x_{n-1})$ *where* Q *is either* $\forall$ *or* $\exists$,

(3d) $\triangle(Qx_0,\cdots,Qx_{n-1})\varphi(x_0,\cdots,x_{n-1})$ *is interpreted as* $(Qx_0,\cdots,Qx_{n-1})\triangle\varphi(x_0,\cdots,x_{n-1})$.

So, for example, the following hold:

- $\triangledown(a\in b)$ is interpreted as $(a\notin b)\,\&\,[(\exists x)(x\overset{\triangledown}{=} a\,\&\,x\in b)\vee(\exists y)(y\overset{\triangledown}{=} b\,\&\,a\in y)]$. However, $y\overset{\triangledown}{=} b\,\&\,a\in y$ implies $a\in b$, and it is impossible because $a\notin b$. So the above is equivalent to $(a\notin b)\,\&\,(\exists x)(x\overset{\triangledown}{=} a\,\&\,x\in b)$.
- $\triangle(a\in b)$ is interpreted as $(a\in b)\vee(\forall x)(x\overset{\triangle}{=} a\vee x\notin b)$ by the similar argument to the above. This means our axiom (A) holds.
- $\triangledown\triangledown(a\in b)$ is interpreted as $\neg\triangledown(a\in b)\,\&\,[(\exists x)(x\overset{\triangledown}{=} a\,\&\,\triangledown(x\in b))\vee(\exists y)(y\overset{\triangledown}{=} b\,\&\,\triangledown(a\in y))]$. So this is interpreted as

$$[a\in b\vee[(\forall x_0)(a\overset{\triangle}{=} x_0\vee x_0\notin b)\,\&\,(\forall x_1)(b\overset{\triangle}{=} x_1\vee a\notin x_1)]]$$
$$\&\,[(\exists x)(x\overset{\triangledown}{=} a\,\&\,(x\notin b\,\&\,((\exists z_0)(z_0\overset{\triangledown}{=} x\,\&\,z_0\in b)\vee(\exists z_1)(z_1\overset{\triangledown}{=} b\,\&\,x\in z_1))))$$
$$\vee(\exists y)(y\overset{\triangledown}{=} b\,\&\,(a\notin y\,\&\,((\exists r_0)(r_0\overset{\triangledown}{=} a\,\&\,r_0\in y)\vee(\exists r_1)(r_1\overset{\triangledown}{=} y\,\&\,a\in r_1))))]$$

As the above, this is equivalent to

$$[a\in b\vee(\forall x_0)(a\overset{\triangle}{=} x_0\vee x_0\notin b)]\,\&\,(\exists x)(x\overset{\triangledown}{=} a\,\&\,(x\notin b\,\&\,((\exists z_0)(z_0\overset{\triangledown}{=} x\,\&\,z_0\in b)))$$

- $\triangledown(a=b)$ is interpreted as $[(\exists x)(x\overset{\triangledown}{=} a\,\&\,x=b)\vee(\exists y)(y\overset{\triangledown}{=} b\,\&\,a=y)]\,\&\,a\neq b$. Clearly this is equivalent to $a\overset{\triangledown}{=} b$: This means axiom (D) holds.
- $\triangle(a=b)$ is interpreted as $[(\forall x)(x\overset{\triangle}{=} a\vee x\neq b)\,\&\,(\forall y)(y\overset{\triangle}{=} b\vee a\neq y)]\vee a=b$ and this means $[(\forall x)(x=b\to x\overset{\triangle}{=} a)\,\&\,(\forall y)(y=a\to y\overset{\triangle}{=} b)]\vee a=b$: Clearly this is equivalent $a\overset{\triangle}{=} b\vee a=b$, so $a\overset{\triangle}{=} b$ holds,
- $\triangle(\varphi(a)\vee\triangledown(a\in b))$ is interpreted as $(\varphi(a)\vee\triangledown(a\in b))\,\&\,[(\forall x)(x\overset{\triangle}{=} a\vee\neg(\varphi(x)\vee\triangledown(a\in b)))]$. This can be interpreted as

$$(\varphi(a)\vee(a\notin b\&(\exists x_0)(x_0\overset{\triangle}{=} a\,\&\,x_0\in b)))\,\&$$
$$(\forall x_1)(x_1\overset{\triangle}{=} a\,\vee\,(\neg\varphi(x_1)\,\&\,(x_1\in b\vee(\forall x_2)(x_1\overset{\triangle}{=} x_2\vee x_2\notin b))))$$

Under this interpretation, it is routine that any of the axiom and axiom schemata is interpreted as being true. This means that the consistency of T implies the consistency of T^*.

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