Comprehension contradicts to the induction within Łukasiewicz predicate logic

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Abstract

We introduce the simpler and shorter proof of Hajek's theorem that the mathematical induction on ω implies a contradiction in the set theory with the comprehension principle within Łukasiewicz predicate logic Ł∀ [H05] by extending the proof in [Y06] so as to be effective in any linearly ordered **MV**-algebra.

1 Introduction

In this paper, we introduce the simpler and shorter proof of Hajek's theorem that the mathematical induction on ω implies a contradiction in the set theory with the comprehension principle within Łukasiewicz predicate logic Ł∀ [H05].

A significance of the set theory with the comprehension principle is to allow a *general form of the recursive definition*: For any formula $\varphi(x, \cdots, y)$, the comprehension principle implies

$$(\exists z)(\forall x)[x \in z \equiv \varphi(x, \cdots, z)]$$

within Grišin logic (classical logic minus the contraction rule) [C03]. This allows us to represent, for example, the set of natural numbers ω, and any partial recursive function on ω.

Let $\mathbf{CŁ}_0$ be a set theory with the comprehension principle within Ł∀, an extension of Grišin logic. $\mathbf{CŁ}_0$ seems to be enough strong to develop an arithmetic: the general form of recursive definition can be used in place of the mathematical induction to define arithmetic. And it had been expected that the arithmetic is a subset of **PA** in classical logic. However, Petr Hajek showed that the following [H05]:

Theorem 1 *The extension* **CL** *of the theory* $\mathbf{CŁ}_0$ *by the(strong) induction scheme on* ω *is contradictory.*

Hajek's result is very surprising, but his proof is very long.

In [Y06], we proved the similar result in a simple way that the induction scheme implies a contradiction in the set theory within ∀**Ł** which is weaker than **Ł**∀. In this paper, we extend this proof so as to be effective in Ł∀.

This theorem shows that the general form of recursive definition contradicts to the induction within **Ł**∀ though they are consistent within classical logic. Therefore $\mathbf{CŁ}_0$ gives a new viewpoint to analyze concepts in arithmetic since it gives a new possibility to give a non-standard arithmetic (an arithmetic developed only by the general form of recursive definition) in a natural way. Since **Ł**∀ is nicely axiomatized, this result might help a study of such recursive definitions.

2 Preliminaries

Our framework in this paper is Łukasiewicz predicate logic $\mathbf{Ł}\forall$. $\mathbf{Ł}\forall$ is a fuzzy logic weaker than $\forall\mathbf{Ł}$, and is axiomatized in Hilbert style as follows.

Definition 1 *The axioms of* $\mathbf{Ł}\forall$ *consists of axioms of propositional Łukasiewicz logic* $\mathbf{Ł}$ *plus the following two additional rules:*

- $\forall x\varphi(x) \to \varphi(t)$,
- $\forall x(\nu \to \varphi) \to (\nu \to (\forall x)\varphi)$ *if* x *is free in* ν.

$\mathbf{Ł}\forall$ proves $\neg\exists\neg\varphi \equiv \forall x\varphi$ and $(\nu \to \exists x\varphi) \to \exists x(\nu \to \varphi)$. We note that $\mathbf{Ł}\forall$ is a predicate logic which is complete for models over **linearly ordered** $\mathbf{M}V$**-algebras**.

Definition 2 *Let* $\mathbf{CŁ}_0$ *be a set theory within* $\mathbf{Ł}\forall$*, which has a binary predicate* $\in$ *and terms of the form* $\{x : \varphi(x)\}$*, and whose axiom scheme is* **the comprehension principle***: for any* φ *not containing* u *freely,* $(\forall u)[u \in \{x : \varphi(x, \cdots)\} \equiv \varphi(u, \cdots)]$.

We can define Leibniz equality $x = y$ iff $(\forall z)[x \in z \leftrightarrow y \in z]$, the empty set $\emptyset = \{x : x \neq x\}$ in standard way.

As we see, $\mathbf{CŁ}_0$ proves the general form of the recursive definition [C03]. In particular, we can construct a term θ such that $\theta =_{\mathrm{ext}} \{u : \varphi(u, \cdots, \theta)\}$ for any formula $\varphi(x, \cdots, y)$. By using this, we can prove that the set of natural numbers ω can be defined as follows:

$$(\forall x)x \in \omega \equiv [x = \emptyset \vee (\exists y)[y \in \omega \wedge x = \{y\}]]$$

For simplicity, we write $n + 1$ instead of $\{n\}$ hereafter.

Once Hajek suggested to introduce the induction scheme:

Definition 3 *The induction scheme on* ω *is a scheme of the form: for any formula* φ,

$$\varphi(0) \wedge (\forall n \in \omega)[\varphi(n) \equiv \varphi(n+1)] \ \textit{infer} \ (\forall x)[x \in \omega \to \varphi(x)]$$

However, Hajek finally proved theorem 1 in a very complex, long proof.

Let $\forall\mathbf{Ł}$ be Łukasiewicz infinite-valued predicate logic whose algebra of truth functions is the standard $\mathbf{M}V$-algebra $[0, 1]_{\mathrm{L}}$ which is generated by $\langle[0, 1], \Rightarrow, *\rangle$. $\forall\mathbf{Ł}$ is stronger than $\mathbf{Ł}\forall$, but $\forall\mathbf{Ł}$ is not recursively axiomatizable. And let $\mathbf{H}$ be the set theory with the comprehension principle within $\forall\mathbf{Ł}$. In [Y06], we proved:

Theorem 2 *The extension of* $\mathbf{H}$ *by the induction scheme on* ω *is contradictory.*

The proof is a very simple, but the proof is only valid for models over Archimedean $\mathbf{M}V$-algebras.

3 A short proof of theorem 1

Here, we extend the proof of the theorem 2 of [Y06]. Let us define

- $\theta = \{\langle n, x\rangle : (n = 0 \ \wedge \ x \notin x) \vee (\exists k \in \omega)[n = k + 1 \ \wedge \ x \in x \to \langle n, x\rangle \in \theta]\}$,
- $R_\omega = \{x : (\exists n)\langle n, x\rangle \in \theta\}$.

The existence of these sets is guaranteed by the recursion theorem. First we can show that $R_\omega \in R_\omega$, i.e. $(\exists n)\langle n, R_\omega\rangle \in \theta$, is provable in **H**:

$$\cfrac{\cfrac{\cfrac{\cfrac{R_\omega \in R_\omega \equiv (\exists n)\,[\langle n, R_\omega\rangle \in \theta]}{R_\omega \in R_\omega \to (\exists n)\,[\langle n, R_\omega\rangle \in \theta]}}{(\exists n)\,[R_\omega \in R_\omega \to \langle n, R_\omega\rangle \in \theta]}}{(\exists n)\,\langle n+1, R_\omega\rangle \in \theta}}{R_\omega \in R_\omega}$$

Let us assume the induction scheme on ω. We remark that the induction scheme implies the crispness of ω [H05]. As we see, $R_\omega \in R_\omega$ is provable, and this means that $\langle 0, R_\omega\rangle \notin \theta$ is provable. For any $n \in \omega$, we can prove $\langle n, R_\omega\rangle \notin \theta \to \langle n+1, R_\omega\rangle \notin \theta$:

$$\cfrac{\cfrac{\cfrac{R_\omega \in R_\omega}{[R_\omega \in R_\omega \to \langle n, R_\omega\rangle \in \theta] \to \langle n, R_\omega\rangle \in \theta}}{\langle n, R_\omega\rangle \notin \theta \to \neg[R_\omega \in R_\omega \to \langle n, R_\omega]}}{\langle n, R_\omega\rangle \notin \theta \to \neg\langle n+1, R_\omega\rangle \in \theta}$$

and $\langle n+1, R_\omega\rangle \notin \theta \to \langle n, R_\omega\rangle \notin \theta$:

$$\cfrac{\cfrac{\cfrac{\langle n+1, R_\omega\rangle \notin \theta}{\neg(R_\omega \in R_\omega \to \langle n, R_\omega\rangle \in \theta)}}{R_\omega \in R_\omega \,\&\, \langle n, R_\omega\rangle \notin \theta}}{\langle n, R_\omega\rangle \notin \theta}$$

Therefore $\langle n, R_\omega\rangle \notin \theta \equiv \langle n+1, R_\omega\rangle \notin \theta$ holds for any $n \in \omega$. The induction scheme proves $(\forall x \in \omega)\langle x, R_\omega\rangle \notin \theta$. $R_\omega \notin R_\omega$ holds by the crispness of ω, but this contradicts to $R_\omega \in R_\omega$.

We note that, this proof involves that the theory **H** is ω-inconsistent, since $\langle j, R_\omega\rangle \notin \theta$ is provable for any standard natural number j though $(\exists x)\langle x, R_\omega\rangle \in \theta$ is provable. Also we note that, since we use $(\varphi \to \exists x\nu) \to \exists x(\varphi \to \nu)$ and double negation elimination, this proof is not valid in some semantics of **BL**$\forall$. $\Box$

4 Conclusion

We introduced the simpler and shorter proof of Hajek's theorem that the mathematical induction on ω implies a contradiction in the set theory with the comprehension principle within **Ł**$\forall$ [H05]. We extended the proof of [Y06] to be effective within **Ł**$\forall$.

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