

AN ALGEBRAIC STUDY OF ASSOCIATION SCHEMES AND ITS APPLICATIONS

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Abstract

This thesis contains two results: new bounds for the sizes of t -designs and the determination of the structure of the coherent configuration on a finite projective geometry. For t -designs, we consider Delsarte's linear programming method for Johnson schemes and provide bounds by using the method and the duality theorem. In addition, we compare our bounds with some other bounds both theoretically and numerically. For the finite projective geometry, we consider the group action on the space, which was well studied by C. F. Dunkl in 1978. We fully use his result and determine the structure of the coherent configuration. As its first application, we determine all irreducible \mathcal{T} -modules for the Terwilliger algebra \mathcal{T} of a Grassmann graph. As its second application, we introduce a linear programming approach to the Erdős–Ko–Rado theorem for singular linear spaces, which was studied by L. Ou, B. Lv and K. Wang in 2014 and check our bounds totally coincide with theirs for small parameters.

Contents

1	Introduction and Preliminaries	1
1.1	Association Schemes	2
1.2	Examples of Association Schemes	6
2	Gelfand Pairs	15
2.1	Equivalent Definitions of Gelfand Pairs	15
2.2	Generalized Theory of Gelfand Pairs	16
3	Delsarte's Linear Programming Method	20
3.1	Delsarte's Linear Programming Bound	20
3.2	Applications to Design Theory	21
4	Finite Projective Geometry	29
4.1	Notations	29
4.2	The Coherent Configuration	29
4.2.1	Parts of the Coherent Configuration	31
4.2.2	Matrices P and Q	35
4.2.3	Values of P with $x = i$ and $u = j$	41
4.3	The Terwilliger Algebras of the Grassmann Graphs	44
4.4	The Erdős-Ko-Rado Theorem for Singular Linear Spaces	48
A	Some Formulas for Binomial Coefficients	50
B	Mathematica Codes	52
B.1	The Bounds in Theorem 3.9	52
B.2	The Linear Programming Bounds in Theorem 4.24	52
C	Tables	55
C.1	Table of Lower Bounds by Theorem 3.9	55
C.2	Table of Upper Bounds by Theorem 4.24	56
	Acknowledgements	85
	Bibliography	86

Chapter 1

Introduction and Preliminaries

Association schemes are important and essential for the study of algebraic combinatorics because they can be considered as a unifying concept of several fields such as coding theory, design theory, algebraic graph theory and finite group theory. This viewpoint of association schemes originates from Delsarte's pioneer dissertation [6] and has been called Delsarte theory after his name. Delsarte theory thereby let us grasp both underlying structures of several fields and connections among them.

In this thesis, we focus on three examples of association schemes: Hamming schemes, Johnson schemes and ones related to a finite projective geometry. Hamming schemes, named after Richard Hamming, are the simple and most important for coding theory. We see them as a warming-up example in Chapter 1. Johnson schemes, named after Selmer M. Johnson, are related to design theory. From the viewpoint of Johnson schemes, we wrestle with the problem about the existence of t -designs with given parameters, that is deeply problematic in design theory. In the last chapter, we study the third example in the more general context of the coherent configuration on the finite projective geometry. The coherent configuration includes as substructures the following well-known association schemes: Grassmann schemes, association schemes based on attenuated spaces and bilinear forms schemes.

In finite group theory, it is important and meaningful to determine their character tables to characterize groups. As this group case, it is also important to determine eigenmatrices, which stand for the character tables in association schemes' language, although it is not easy. For our examples, the eigenmatrices of Hamming schemes and Johnson schemes were first determined by P. Delsarte in his dissertation. We will see them in Chapter 1. For the last example, we will determine the "eigenmatrices" in Chapter 4. Note that, for some special cases of our last example namely bilinear forms schemes and association schemes based on attenuated spaces, their eigenmatrices have already been determined by Delsarte [7] and Kurihara [17].

Next, we will closely introduce two main topics in this thesis: block design theory and finite projective geometry. The study of block designs dates back to nineteenth century. In 1850, Thomas P. Kirkman proposed the following problem (cf. Kirkman [15]):

A school mistress has fifteen girl pupils and she wishes to take them on a daily walk. The girls are to walk in five rows of three girls each. It is required that no two girls should walk in the same row more than once per week. Can this be done?

This problem is generally known as "Kirkman's schoolgirl problem". In modern terms of block designs, the problem is equivalent to the construction of a certain $2-(15, 3, 1)$ design. In another word, this is one of the earliest examples of the concept of a t - (v, k, λ) design (t -design). We see the definition of a t -design in Chapter 3. For given parameters t, v, k, λ , it is still difficult to check for the existence of a t - (v, k, λ) design today. In 1973, Delsarte [6] came up with a new approach, called the linear programming method, to this question with association schemes. The linear programming problems produce lower bounds for the parameter λ from given parameters t, v, k . The bounds were much better than some other bounds which had been known at that time. However, we point out one serious problem of the linear programming method. That is in difficulty of computing with extremely large parameters because of technical aspects. Therefore we construct new lower bounds which

are explicit functions in t, v, k and do not require solving any linear programming problems (Theorem 3.9). Our proof is based on Delsarte's linear programming method and the duality theorem that is a well-known theorem in the field of linear programming. Our bounds are excellent not only in the computing time but also in the contribution to the existence question because our lower bounds are at least as good as the Fisher bounds, which are well-known superior lower bounds.

The study of association schemes related to a finite projective geometry traces back to Delsarte [7]. In 1978, P. Delsarte constructed association schemes on the set of bilinear forms on a pair of finite dimensional vector spaces over a finite field and determined the structures in the same paper. He viewed the bilinear forms schemes as q -analogs of Hamming schemes. In 2010, Wang, Guo and Li constructed association schemes based on attenuated spaces. (cf. Wang–Guo–Li [29].) Since attenuated spaces contain as a special case the set of bilinear forms, the association schemes can be seen as a generalization of bilinear forms schemes. In 2013, Kurihara [17] determined the structures of the association schemes based on attenuated spaces. With these previous papers in mind, my attention is drawn to the fact that attenuated spaces are substructures of the finite projective geometry and to the construction of a structure based on the finite projective geometry. To construct, we consider a group action on the finite projective geometry. Since this group action is much complicated and have been studied well by Dunkl [8], we use his results as useful tools of our calculations. As applications, we determine the Terwilliger algebras of Grassmann graphs and give a new approach to the Erdős–Ko–Rado theorem for singular linear spaces.

The Terwilliger algebras of Q -polynomial distance-regular graphs are introduced by Terwilliger [26, 27, 28] and generalized by Suzuki [22]. Schrijver's semidefinite programming method (cf. Schrijver [21]) is one of applications of Terwilliger algebras. In his paper [28], P. Terwilliger makes a list of data about the irreducible representations of the original Terwilliger algebras of many families of Q -polynomial distance-regular graphs without any proofs. However, rigorous proofs have not been given to his data (as published papers) even today, with a few exceptions. We describe all the irreducible representations of the Terwilliger algebras of Grassmann graphs and provide the details of our calculations. We discuss some special cases of Suzuki's generalization in a unified manner, so our results are indeed new. We have found that the Terwilliger algebras have interestingly "nice" characters.

The Erdős–Ko–Rado theorem was originally proposed by Erdős, Ko and Rado in 1961. (Erdős–Ko–Rado [9]) The theorem is as follows. Consider a family \mathcal{F} of d -subsets of a v -set such that any two elements $x, y \in \mathcal{F}$ satisfy the condition $|x \cap y| \geq t$. Then the maximum size of \mathcal{F} is $\binom{v-t}{d-t}$ if v is large enough. In 1984, Wilson [30] gave another proof of it from an association schemes' viewpoint. Specifically, he used Delsarte's linear programming method. Since then, there are various kinds of derived Erdős–Ko–Rado theorems and their proofs by using association schemes. For example, Tanaka [23] gave a proof of the Erdős–Ko–Rado theorem for Grassmann graphs by using Delsarte's linear programming method. In this thesis, we consider the Erdős–Ko–Rado theorem for singular linear spaces studied by Ou, Lv and Wang in 2014. (cf. Ou–Lv–Wang [19].) Their proof is not from an association schemes' viewpoint but from a combinatorial one. Moreover their proof requires some restrictions on the parameters. We propose a linear programming approach to the Erdős–Ko–Rado theorem which hopefully will lead to a proof of the theorem in full generality. In addition, we found that our computational results agree with the result by L. Ou, B. Lv and K. Wang.

The rest of this chapter provides the required preliminary knowledge before proceeding on to the results. The contents are based on the following: Bannai–Ito [2], Brouwer–Haemers [4], Delsarte [6] and Godsil [11].

1.1 Association Schemes

An association scheme with d classes based on a finite set X is a pair of X and relations $\{R_i\}_{i=0}^d$ on X such that

- (i) $\{R_i\}_{i=0}^d$ is a partition of $X \times X$;
- (ii) $R_0 = \{(x, x) \mid x \in X\}$;

- (iii) For all $0 \leq i \leq d$, there exists an integer i' such that $\{(y, x) \mid (x, y) \in R_i\} = R_{i'}$;
- (iv) For all $0 \leq k \leq d$, the number $p_{i,j}^k = |\{z \in X \mid (x, z) \in R_i, (z, y) \in R_j\}|$ is independent of the choice of $(x, y) \in R_k$.

The numbers $p_{i,j}^k$ are called the intersection numbers of the association scheme. An association scheme is symmetric if the condition (iii) is replaced by:

- (iii') For all $0 \leq i \leq d$, $\{(y, x) \mid (x, y) \in R_i\} = R_i$.

A commutative association scheme requires conditions (i), (ii), (iii), (iv) and

- (v) $p_{i,j}^k = p_{j,i}^k$ for all $0 \leq i, j, k \leq d$.

We can see (X, R_j) as a simple directed regular graph. Each valency $n_j = |\{z \in X \mid (x, z) \in R_j\}|$ of the regular graph is called the valency of the association scheme. The adjacency matrix of the graph (X, R_j) is denoted by A_j . We will denote by I the identity matrix and by J the all-one matrix. All the conditions of an association scheme above can be restated in terms of the adjacency matrices as follows:

- (i) $\sum_{i=0}^d A_i = J$,
- (ii) $A_0 = I$,
- (iii) For all $0 \leq i \leq d$, there exists an integer i' such that $A_i^T = A_{i'}$,
- (iii') For all $0 \leq i \leq d$, A_i is symmetric,
- (iv) There exist the numbers $p_{i,j}^k$ such that $A_i A_j = \sum_{k=0}^d p_{i,j}^k A_k$,
- (v) $p_{i,j}^k = p_{j,i}^k$ for all $0 \leq i, j, k \leq d$.

Note that a symmetric association scheme is commutative, i.e. the condition (iii') implies (iii) and (v).

Let \mathcal{A} be the subalgebra of $\text{Mat}_X(\mathbb{C})$ spanned by the adjacency matrices $\{A_i\}_{i=0}^d$, where $\text{Mat}_X(\mathbb{C})$ is the full matrix algebra indexed by X over \mathbb{C} . \mathcal{A} is called the adjacency algebra of the association scheme or the Bose–Mesner algebra of the association scheme.

In Chapter 4, we consider the concept of coherent configurations which is less conditional and defined as follows. A pair $(X, \{R_i\}_{i=0}^d)$ is called a coherent configuration if it satisfies the conditions (i), (iii), (iv) and

- (ii') There exists $I \subset \{0, 1, \dots, d\}$ such that $\bigcup_{i \in I} R_i = \{(x, x) \mid x \in X\}$.

See Higman [12, 13] for more information. He calls an association scheme a homogeneous coherent configuration. The next theorem will be referred in Chapter 4.

Theorem 1.1. *Let G be a permutation group acting on a finite set X . We denote by R_0, R_1, \dots, R_d to be totality of G -orbits, where we consider G acting on the Cartesian product $X \times X$. Then the pair $(X, \{R_i\}_{i=0}^d)$ forms a coherent configuration.*

Proof. We show that it satisfies all the conditions.

- (i) This is obvious from the definition of R_0, R_1, \dots, R_d .
- (ii') This is because the set $\{(x, x) \mid x \in X\}$ is invariant under the G -action.
- (iii) For any i , the set $\{(y, x) \mid (x, y) \in R_i\}$ is again a G -orbit.
- (iv) For any i, j, k , the G -action preserves the number $|\{z \in X \mid (x, z) \in R_i, (z, y) \in R_j\}|$ for any $(x, y) \in R_k$.

□

For the rest of this section, we only consider commutative association schemes. Let $(X, \{R_i\}_{i=0}^d)$ be a commutative association scheme. By conditions (iii), (iv) and (v), the adjacency matrices $\{A_i\}_{i=0}^d$ are pairwise commutative normal matrices.

Theorem 1.2. *Let \mathcal{A} be the adjacency algebra of a commutative association scheme $(X, \{R_i\}_{i=0}^d)$. Then \mathcal{A} has a unique basis of primitive idempotents E_0, E_1, \dots, E_d .*

Proof. Since A_i ($0 \leq i \leq d$) are normal matrices, we have the spectral decomposition

$$A_i = \sum_j \theta_{i,j} Y_{i,j},$$

where the $\theta_{i,j}$ are the distinct eigenvalues of A_i and each $Y_{i,j}$ is the orthogonal projection onto the eigenspace corresponding to $\theta_{i,j}$. Then

$$I = I^{d+1} = \prod_{i=0}^d I = \prod_{i=0}^d \left(\sum_j Y_{i,j} \right) = \sum_{j_0, j_1, \dots, j_d} Y_{0,j_0} Y_{1,j_1} \cdots Y_{d,j_d}.$$

Let $E_0, E_1, \dots, E_{d'}$ be the nonzero summands in the right-hand side, i.e. $I = \sum_{k=0}^{d'} E_k$. Since the E_k are products of the orthogonal projections $Y_{i,j}$, they are also pairwise orthogonal idempotents. In order to show that the idempotents E_k are in the adjacency algebra \mathcal{A} , we check that the orthogonal projections $Y_{i,j}$ are polynomials in A_i :

$$Y_{i,j} = \prod_{k \neq j} \frac{A_i - \theta_{i,k} I}{\theta_{i,j} - \theta_{i,k}}.$$

It is because we have

$$\left(\prod_{k \neq j} \frac{A_i - \theta_{i,k} I}{\theta_{i,j} - \theta_{i,k}} \right) Y_{i,j} = \prod_{k \neq j} \frac{\theta_{i,j} Y_{i,j} - \theta_{i,k} Y_{i,j}}{\theta_{i,j} - \theta_{i,k}} = \left(\prod_{k \neq j} \frac{\theta_{i,j} - \theta_{i,k}}{\theta_{i,j} - \theta_{i,k}} \right) Y_{i,j} = Y_{i,j}.$$

Thus the primitive idempotents E_k are also in the adjacency algebra \mathcal{A} .

At last, we show that $E_0, E_1, \dots, E_{d'}$ form a basis of the adjacency algebra \mathcal{A} . Since they are linearly independent and in the adjacency algebra \mathcal{A} , it is enough to show $d \leq d'$. Now, for $0 \leq i \leq d$, we have

$$A_i = A_i I = A_i \left(\sum_{j_0, j_1, \dots, j_d} Y_{0,j_0} Y_{1,j_1} \cdots Y_{d,j_d} \right) = \sum_{j_0, j_1, \dots, j_d} \theta_{i,j_i} Y_{0,j_0} Y_{1,j_1} \cdots Y_{d,j_d} \in \langle E_0, E_1, \dots, E_{d'} \rangle.$$

Therefore we have $d \leq d'$, which is our desired result. \square

Since the matrix $|X|^{-1}J$ is a primitive idempotent, we shall take $E_0 = |X|^{-1}J$. Now, there exist two bases for the adjacency algebra \mathcal{A} . The change of basis matrix for converting from $\{E_i\}$ to $\{A_i\}$ is denoted by P , and the other side will be denoted by Q . These matrices are called the first and the second eigenmatrices respectively;

$$A_j = \sum_{i=0}^d P_{i,j} E_i, \quad E_i = \frac{1}{|X|} \sum_{j=0}^d Q_{j,i} A_j. \quad (1.1)$$

Then $|X|^{-1}Q$ is the inverse matrix of P and $A_j E_i = P_{i,j} E_i$ for all $0 \leq i, j \leq d$. In other words, $\{P_{i,j}\}_{i=0}^d$ is the set of all eigenvalues of A_j and the columns of the E_i s are corresponding eigenvectors. Each multiplicity $\mu_i = \text{rank } E_i$ of eigenvalue $P_{i,j}$ of A_j is called the multiplicity of the commutative association scheme.

Theorem 1.3. *Let P, Q be the first and second eigenmatrices of a commutative association scheme, and let $\{\mu_i\}_{i=0}^d, \{n_j\}_{j=0}^d$ be the multiplicities and the valencies of it, respectively. Then for all $0 \leq i, j \leq d$, we have*

$$\mu_i P_{i,j} = n_j Q_{j,i}.$$

Proof. We consider the trace of $A_j E_i$ in two ways,

$$\begin{aligned}\mathrm{Tr}(A_j E_i) &= \frac{1}{|X|} \sum_{k=0}^d Q_{k,i} \mathrm{Tr}(A_j A_k) = \frac{1}{|X|} \sum_{k=0}^d Q_{k,i} \left(\sum_{l=0}^d p_{j,k}^l \mathrm{Tr}(A_l) \right) = \sum_{k=0}^d Q_{k,i} p_{j,k}^0 = Q_{j',i} n_{j'}, \\ \mathrm{Tr}(A_j E_i) &= \sum_{k=0}^d P_{k,j} \mathrm{Tr}(E_k E_i) = P_{i,j} \mathrm{Tr}(E_i) = \mu_i P_{i,j}.\end{aligned}$$

Here, we use the property that $p_{j,k}^0 = |\{z \in X \mid (x,z) \in R_j, (z,x) \in R_k\}| = \delta_{j',k} n_{j'}$ for any $x \in X$. \square

This theorem and the fact that the first and second matrices P, Q are orthogonal to each other imply that the rows (or columns) of each of the matrices are also orthogonal.

The adjacency algebra \mathcal{A} is closed under ordinary matrix multiplication and componentwise multiplication which is denoted by \circ . Then $\{A_i\}_{i=0}^d$ is the basis of primitive idempotents with respect to this multiplication. Then $E_i \circ E_j$ is a linear combination of E_0, E_1, \dots, E_d and write

$$E_i \circ E_j = \frac{1}{|X|} \sum_{k=0}^d q_{i,j}^k E_k. \quad (1.2)$$

The numbers $q_{i,j}^k$ are called the Krein parameters of the commutative association scheme. The following relations for these parameters are well known and we will need them later on.

Theorem 1.4. *Let Q be the second eigenmatrix of a symmetric association scheme, and let $\{\mu_i\}_{i=0}^d, \{q_{i,j}^k\}_{i,j,k=0}^d$ be the multiplicities and the Krein parameters of it, respectively. Then for all $0 \leq i, j, k \leq d$, we have*

- (i) $Q_{0,i} = \mu_i$,
- (ii) $q_{i,j}^0 = \delta_{i,j} \mu_j$,
- (iii) $Q_{i,j} Q_{i,k} = \sum_{l=0}^d q_{j,k}^l Q_{i,l}$.
- (iv) *There exists at least one negative entry in every column of Q except the 0-th column.*

Proof. (i) Since A_j has 0 on diagonal entries if $j \neq 0$ and $A_0 = I$, we have

$$\mu_i = \mathrm{Tr} E_i = \frac{1}{|X|} \sum_{j=0}^d Q_{j,i} \mathrm{Tr} A_j = \frac{1}{|X|} Q_{0,i} \mathrm{Tr} A_0 = Q_{0,i}.$$

- (ii) We consider the sum of all entries of $E_i \circ E_j$ in two ways,

$$\begin{aligned}\sum_{x,y \in X} (E_i \circ E_j)_{x,y} &= \frac{1}{|X|} \sum_{k=0}^d q_{i,j}^k \sum_{x,y \in X} (E_k)_{x,y} = \frac{1}{|X|} \sum_{k=0}^d q_{i,j}^k \sum_{x,y \in X} (E_k E_0)_{x,y} = q_{i,j}^0, \\ \sum_{x,y \in X} (E_i \circ E_j)_{x,y} &= \sum_{x,y \in X} (E_i)_{x,y} (E_j)_{x,y} = \mathrm{Tr}(E_i E_j^T) = \mathrm{Tr}(E_i E_j) = \delta_{i,j} \mu_j.\end{aligned}$$

- (iii) We consider $A_i \circ E_j \circ E_k$ in two ways,

$$\begin{aligned}A_i \circ E_j \circ E_k &= A_i \circ \left(\frac{1}{|X|} \sum_{l=0}^d Q_{l,j} A_l \right) \circ E_k = \frac{1}{|X|} Q_{i,j} (A_i \circ E_k) = \frac{1}{|X|^2} Q_{i,j} Q_{i,k} A_i, \\ A_i \circ E_j \circ E_k &= A_i \circ \left(\frac{1}{|X|} \sum_{l=0}^d q_{j,k}^l E_l \right) = \frac{1}{|X|^2} \sum_{l=0}^d q_{j,k}^l Q_{i,l} A_i.\end{aligned}$$

(iv) All entries in the 0-th column are 1 since

$$E_0 = |X|^{-1}J = |X|^{-1}(A_0 + A_1 + \cdots + A_d).$$

Let P be the first eigenmatrix of the association scheme, and $\{n_j\}_{j=0}^d$ be the valencies of it. Then, for $0 \leq j \leq d$ we have

$$n_j J = A_j J = \left(\sum_{i=0}^d P_{i,j} E_i \right) J = P_{0,j} J.$$

Thus $P_{0,j} = n_j$. For any integers $1 \leq i \leq d$, we have

$$0 = (PQ)_{0,i} = \sum_{j=0}^d P_{0,j} Q_{j,i} = \sum_{j=0}^d n_j Q_{j,i}.$$

Since the n_j are positive, there exists at least one index j such that $Q_{j,i} < 0$. \square

1.2 Examples of Association Schemes

In this section, we will give two examples of association schemes, Hamming schemes and Johnson schemes. Both of them are symmetric association schemes.

Let \mathcal{Q} be a set of cardinality q . For a positive integer n , we consider the n -th Cartesian power $X = \mathcal{Q}^n$. The Hamming distance between two points $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ of X is

$$d_H(x, y) = |\{i \mid x_i \neq y_i, 1 \leq i \leq n\}|. \quad (1.3)$$

Let R_i be the i -th distance relation on X . Then $(X, \{R_i\}_{i=0}^n)$ is a symmetric association scheme, the Hamming scheme $H(n, q)$.

Theorem 1.5. *Let P, Q be the first and second eigenmatrices of the Hamming scheme $H(n, q)$, and let $\{\mu_i\}_{i=0}^n, \{n_j\}_{j=0}^n$ be the multiplicities and the valencies of it respectively. Then for $0 \leq i, j \leq n$,*

$$\begin{aligned} P_{i,j} = Q_{i,j} = K_j(i) &= \sum_{k=0}^j (-1)^k (q-1)^{j-k} \binom{i}{k} \binom{n-i}{j-k}, \\ \mu_i = n_i &= \binom{n}{i} (q-1)^i. \end{aligned}$$

Here $K_j(x)$ is a so-called Krawtchouk polynomial and it has degree j in the indeterminate x .

Proof of Theorem 1.5. For the proof, we identify the set \mathcal{Q} as an abelian group of order q and set $\mathbf{0} = (0, \dots, 0) \in X$ for the identity element $0 \in \mathcal{Q}$. For an integer $0 \leq j \leq n$, let $S_j \subset X$ be the set of all points whose distances from the point $\mathbf{0} \in X$ are exactly j . Then the adjacency matrix of the Cayley graph $\text{Cay}(X, S_j)$ coincides with the j -th adjacency matrix A_j of the Hamming scheme $H(n, q)$. From the theorem in [25, Theorem 1, Chapter 13], all eigenvalues of the adjacency matrix of $\text{Cay}(X, S_j)$ are

$$\mathcal{F}\delta_{S_j}(\chi) = \sum_{a \in S_j} \chi(a) \quad \chi \in \hat{X},$$

where \mathcal{F} denotes the discrete Fourier transform and \hat{X} the dual group of X . Let $\chi = \chi_1 \otimes \chi_2 \otimes \cdots \otimes \chi_n$, where each χ_i is a character of \mathcal{Q} . We claim that $\mathcal{F}\delta_{S_j}(\chi)$ is equal to the Krawtchouk polynomial $K_j(i)$, where $i = |\{l \mid \chi_l \neq 1, 1 \leq l \leq n\}|$.

$$\mathcal{F}\delta_{S_j}(\chi) = \sum_{a \in S_j} \chi(a) = \sum_{(a_1, a_2, \dots, a_n) \in S_j} \left(\prod_{l=1}^n \chi_l(a_l) \right)$$

$$\begin{aligned}
&= \sum_{\substack{J \subset \{1, 2, \dots, n\} \\ |J|=j}} \sum_{\substack{(a_1, a_2, \dots, a_n) \in X \\ a_i \neq 0 \text{ iff } i \in J}} \left(\prod_{l=1}^n \chi_l(a_l) \right) \\
&= \sum_{\substack{J \subset \{1, 2, \dots, n\} \\ |J|=j}} \sum_{\substack{(a_1, a_2, \dots, a_n) \in X \\ a_i \neq 0 \text{ iff } i \in J}} \left(\prod_{l \in J} \chi_l(a_l) \right) \\
&= \sum_{\substack{J \subset \{1, 2, \dots, n\} \\ |J|=j}} \prod_{l \in J} \left(\sum_{a \in \mathcal{Q} \setminus \{0\}} \chi_l(a) \right) \\
&= \sum_{\substack{J \subset \{1, 2, \dots, n\} \\ |J|=j}} \prod_{l \in J} \left(\sum_{a \in \mathcal{Q}} \chi_l(a) - 1 \right) \\
&= \sum_{k=0}^j \sum_{\substack{J \subset \{1, 2, \dots, n\} \\ |J|=j \\ |\{l \in J | \chi_l \neq 1\}|=k}} \prod_{l \in J} \left(\sum_{a \in \mathcal{Q}} \chi_l(a) - 1 \right) \\
&= \sum_{k=0}^j \sum_{\substack{J \subset \{1, 2, \dots, n\} \\ |J|=j \\ |\{l \in J | \chi_l \neq 1\}|=k}} (-1)^k (q-1)^{j-k} \\
&= \sum_{k=0}^j (-1)^k (q-1)^{j-k} \binom{i}{k} \binom{n-i}{j-k}.
\end{aligned}$$

Thus the i -th eigenvalue of A_j , which is in other words the (i, j) -entry of the first eigenmatrix P , is given by the Krawtchouk polynomial $K_j(i)$.

In addition, there are formulas for Krawtchouk polynomials called the orthogonality relation and Askey–Wilson duality [18, Theorems 16 & 17, Chapter 5]:

$$\sum_{i=0}^n \binom{n}{i} (q-1)^i K_r(i) K_s(i) = q^n \binom{n}{s} (q-1)^s \delta_{r,s}, \quad (1.4)$$

$$\binom{n}{i} (q-1)^i K_j(i) = \binom{n}{j} (q-1)^j K_i(j). \quad (1.5)$$

These two equations imply the following:

$$\begin{aligned}
(|X|^{-1} P^2)_{r,s} &= \sum_{i=0}^n q^{-n} K_i(r) K_s(i) \\
&= \sum_{i=0}^n q^{-n} \frac{\binom{n}{i} (q-1)^i}{\binom{n}{r} (q-1)^r} K_r(i) K_s(i) \quad \text{by (1.5)} \\
&= \delta_{r,s} \quad \text{by (1.4).}
\end{aligned}$$

This implies that the second eigenmatrix Q is the same as P .

Finally we have, for some $x \in X$,

$$n_i = |\{y \in X \mid d_H(x, y) = i\}| = \binom{n}{i} (q-1)^i.$$

Then (1.5) and Theorem 1.3 tell us that $n_i = \mu_i$. □

Let \mathcal{S} be a set of cardinality v . For a positive integer k , define \mathcal{S}_k to be the set of all subsets of size k . Then we consider the set $X = \mathcal{S}_k$. The Johnson distance between two subsets x, y of X is

$$d_J(x, y) = k - |x \cap y|. \quad (1.6)$$

Let R_i be the i -th distance relation on X . Then $(X, \{R_i\}_{i=0}^k)$ is a symmetric association scheme, the Johnson scheme $J(v, k)$.

Define the matrix B_i ($0 \leq i \leq k$) whose rows are indexed by the vertex set $X = \mathcal{S}_k$ and whose columns are indexed by \mathcal{S}_i such that

$$(B_i)_{x,z} = \begin{cases} 1 & \text{if } z \subset x, \\ 0 & \text{otherwise,} \end{cases} \quad x \in X, z \in \mathcal{S}_i. \quad (1.7)$$

Lemma 1.6. *Let B_i be the matrix defined above. Set $C_i := B_i B_i^T$ ($0 \leq i \leq k$).*

- (i) *For any $0 \leq i \leq k$, we have $C_i = \sum_{l=i}^k \binom{l}{i} A_{k-l}$.*
- (ii) *For any $0 \leq r, s \leq k$, we have $C_r C_s = \sum_{t=0}^{\min\{r,s\}} \binom{k-t}{r-t} \binom{k-t}{s-t} \binom{v-r-s}{v-k-t} C_t$.*
- (iii) *There exists an order of the primitive idempotents such that $C_i \in \langle E_0, E_1, \dots, E_i \rangle$.*
- (iv) *For any $0 \leq i \leq k$, we have $\text{rank } C_i = \text{rank } B_i = \binom{v}{i}$.*

Proof. (i) By definition, we have

$$(C_i)_{x,y} = \sum_{z \in \mathcal{S}_i} (B_i)_{x,z} (B_i)_{y,z} = |\{z \in \mathcal{S}_i \mid z \subset (x \cap y)\}| = \binom{l}{i},$$

where $d_J(x, y) = k - l$. This is the coefficient of A_{k-l} .

- (ii) We count the (x, y) -entry of the matrix $C_r C_s$ in the following way. Here we set $u = |x \cap y|$.

$$\begin{aligned} & (C_r C_s)_{x,y} \\ &= \sum_{z \in X} |\{\xi \in \mathcal{S}_r \mid \xi \subset (x \cap z)\}| \times |\{\eta \in \mathcal{S}_s \mid \eta \subset (z \cap y)\}| \\ &= |\{(\xi, \eta, z) \in \mathcal{S}_r \times \mathcal{S}_s \times X \mid \xi \subset x, \eta \subset y, (\xi \cup \eta) \subset z\}| \\ &= \sum_{\substack{\xi \in \mathcal{S}_r \\ \xi \subset x}} \sum_{\substack{\eta \in \mathcal{S}_s \\ \eta \subset y}} |\{z \in X \mid (\xi \cup \eta) \subset z\}| \\ &= \sum_{j=0}^{\min\{r,s\}} \sum_{\substack{\xi \in \mathcal{S}_r \\ \xi \subset x}} \sum_{\substack{\eta \in \mathcal{S}_s \\ \eta \subset y \\ |\xi \cap \eta| = j}} |\{z \in X \mid (\xi \cup \eta) \subset z\}| \\ &= \sum_{j=0}^{\min\{r,s\}} \sum_{\substack{\xi \in \mathcal{S}_r \\ \xi \subset x}} \sum_{\substack{\eta \in \mathcal{S}_s \\ \eta \subset y \\ |\xi \cap \eta| = j}} \binom{v-r-s+j}{v-k} \\ &= \sum_{j=0}^{\min\{r,s\}} \sum_{i=0}^r \sum_{\substack{\xi \in \mathcal{S}_r \\ \xi \subset x}} \sum_{\substack{\eta \in \mathcal{S}_s \\ \eta \subset y \\ |\xi \cap y| = i \\ |\xi \cap \eta| = j}} \binom{v-r-s+j}{v-k} \\ &= \sum_{j=0}^{\min\{r,s\}} \sum_{i=0}^r \sum_{\substack{\xi \in \mathcal{S}_r \\ \xi \subset x \\ |\xi \cap y| = i}} \binom{k-i}{s-j} \binom{i}{j} \binom{v-r-s+j}{v-k} \\ &= \sum_{j=0}^{\min\{r,s\}} \sum_{i=0}^r \binom{k-u}{r-i} \binom{u}{i} \binom{k-i}{s-j} \binom{i}{j} \binom{v-r-s+j}{v-k} \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=0}^{\min\{r,s\}} \sum_{i=0}^r \left(\sum_{t=i}^u (-1)^{t-i} \binom{t}{i} \binom{k-t}{r-t} \binom{u}{t} \right) \\
&\quad \times \binom{k-i}{s-j} \binom{i}{j} \binom{v-r-s+j}{v-k} \quad \text{by Theorem A.2 (ii)} \\
&= \sum_{t=0}^u \sum_{j=0}^{\min\{r,s\}} \left(\sum_{i=0}^t (-1)^{t-i} \binom{i}{j} \binom{k-i}{s-j} \binom{t}{i} \right) \\
&\quad \times \binom{k-t}{r-t} \binom{u}{t} \binom{v-r-s+j}{v-k} \\
&= \sum_{t=0}^{\min\{r,s\}} \sum_{j=0}^{\min\{r,s\}} (-1)^{t-j} \binom{k-t}{r-t} \binom{u}{t} \binom{v-r-s+j}{v-k} \\
&\quad \times \left(\sum_{i=j}^t (-1)^{i-j} \binom{i}{j} \binom{k-i}{k-s+j-i} \binom{t}{i} \right) \\
&= \sum_{t=0}^{\min\{r,s\}} \sum_{j=0}^{\min\{r,s\}} (-1)^{t-j} \binom{k-t}{r-t} \binom{u}{t} \binom{v-r-s+j}{v-k} \\
&\quad \times \binom{k-t}{k-s} \binom{t}{j} \quad \text{by Theorem A.2 (ii)} \\
&= \sum_{t=0}^{\min\{r,s\}} \left(\sum_{j=0}^{\min\{r,s\}} (-1)^{t-j} \binom{t}{j} \binom{v-r-s+j}{v-k} \right) \\
&\quad \times \binom{k-t}{k-s} \binom{k-t}{r-t} \binom{u}{t} \\
&= \sum_{t=0}^{\min\{r,s\}} \left(\sum_{j=0}^t (-1)^{t-j} \binom{t}{j} \binom{v-r-s+j}{v-k} \right) \\
&\quad \times \binom{k-t}{k-s} \binom{k-t}{r-t} \binom{u}{t} \\
&= \sum_{t=0}^{\min\{r,s\}} \left(\binom{v-r-s}{v-k-t} \binom{k-t}{k-s} \binom{k-t}{r-t} \binom{u}{t} \right) \quad \text{by Theorem A.2 (i)} \\
&= \sum_{t=0}^{\min\{r,s\}} \left(\binom{v-r-s}{v-k-t} \binom{k-t}{k-s} \binom{k-t}{r-t} (C_t)_{x,y} \right).
\end{aligned}$$

(iii) The equation (ii) implies the chain of ideals

$$\langle C_0 \rangle \subsetneq \langle C_0, C_1 \rangle \subsetneq \langle C_0, C_1, C_2 \rangle \subsetneq \cdots \subsetneq \langle C_0, C_1, \dots, C_k \rangle = \mathcal{A}.$$

We define the order of the primitive idempotents of \mathcal{A} such that $\langle E_0, E_1, \dots, E_r \rangle = \langle C_0, C_1, \dots, C_r \rangle$ for all $0 \leq r \leq k$. Note that we have $E_0 \in \langle C_0 \rangle$ by the definition of C_0 in (i). In other words, C_r is a linear combination of E_0, E_1, \dots, E_r .

(iv) Since $C_i = B_i B_i^T$, we have $\text{rank } C_i = \text{rank}(B_i B_i^T) = \text{rank } B_i$. We claim that B_i has full rank which is a result due to Kantor [14]. Then we have $\text{rank } B_i = |\mathcal{S}_i| = \binom{v}{i}$.

Fix a subset $z \in \mathcal{S}_i$ and $x_0, x_1, \dots, x_{i-1} \in X$ such that for any $0 \leq s \leq i-1$, $|x_s \cap z| = s$. We consider the $i \times i$ matrix T defined by

$$T_{r,s} = \sum_{\substack{y \in \mathcal{S}_i \\ |y \cap z| = r}} (B_i)_{x_s, y} = |\{y \in \mathcal{S}_i \mid |y \cap z| = r, y \subset x_s\}|.$$

Note that $T_{r,s} = 0$ if $s < r$ and $T_{r,r} \neq 0$. This implies that T is invertible.

Next we consider $\Gamma < \text{Sym}(X)$ which is the stabilizer subgroup of the fixed subset $z \in \mathcal{S}_i$ in the symmetric group $\text{Sym}(X)$:

$$\Gamma = \{\alpha \in \text{Sym}(X) \mid z^\alpha = z\},$$

where z^α denotes the image of z under α -action. Note that $(B_i)_{x^\alpha, y^\alpha} = (B_i)_{x, y}$ because $y \subset x$ implies $y^\alpha \subset x^\alpha$.

For later use, we determine the number $|\{\alpha \in \Gamma \mid y^\alpha = y'\}|$ for some $y, y' \in \mathcal{S}_i$ with $|y \cap z| = |y' \cap z| = r$. By the assumption $|y \cap z| = |y' \cap z|$, there exist $\beta \in \Gamma$ such that $y^\beta = y'$.

$$\begin{aligned} |\{\alpha \in \Gamma \mid y^\alpha = y'\}| &= |\{\alpha \in \Gamma \mid y^{\alpha\beta^{-1}} = y\}| \\ &= |\{\alpha \in \Gamma \mid y^\alpha = y\}| \\ &= |\{\alpha \in \text{Sym}(X) \mid y^\alpha = y, z^\alpha = z\}| \\ &= |\{\alpha \in \text{Sym}(X) \mid (y \cap z)^\alpha = (y \cap z), y^\alpha = y, z^\alpha = z\}| \\ &= r! \times \{(i - r)!\}^2 \times (v - 2i + r)! . \end{aligned}$$

To prove that the rank of B_i is $\binom{v}{i}$, we suppose that the column vectors of B_i are linearly dependent, i.e. there exist $a_y \in \mathbb{R}$ ($y \in \mathcal{S}_i \setminus \{z\}$) such that for any $x \in X$,

$$(B_i)_{x, z} = \sum_{\substack{y \in \mathcal{S}_i \\ y \neq z}} a_y (B_i)_{x, y}.$$

For $\alpha \in \Gamma$, we have

$$(B_i)_{x, z} = (B_i)_{x^\alpha, z} = \sum_{\substack{y \in \mathcal{S}_i \\ y \neq z}} a_y (B_i)_{x^\alpha, y} = \sum_{\substack{y \in \mathcal{S}_i \\ y \neq z}} a_{y^\alpha} (B_i)_{x^\alpha, y^\alpha} = \sum_{\substack{y \in \mathcal{S}_i \\ y \neq z}} a_{y^\alpha} (B_i)_{x, y}.$$

Then

$$\begin{aligned} |\Gamma|(B_i)_{x, z} &= \sum_{\alpha \in \Gamma} (B_i)_{x, z} \\ &= \sum_{\alpha \in \Gamma} \sum_{\substack{y \in \mathcal{S}_i \\ y \neq z}} a_{y^\alpha} (B_i)_{x, y} \\ &= \sum_{\alpha \in \Gamma} \sum_{r=0}^{i-1} \sum_{\substack{y \in \mathcal{S}_i \\ |y \cap z|=r}} a_{y^\alpha} (B_i)_{x, y} \\ &= \sum_{r=0}^{i-1} \sum_{\substack{y \in \mathcal{S}_i \\ |y \cap z|=r}} \sum_{\substack{y' \in \mathcal{S}_i \\ |y' \cap z|=r}} \sum_{\substack{\alpha \in \Gamma \\ y^\alpha = y'}} a_{y'} (B_i)_{x, y} \\ &= \sum_{r=0}^{i-1} \left(b_r \sum_{\substack{y' \in \mathcal{S}_i \\ |y' \cap z|=r}} a_{y'} \right) \sum_{\substack{y \in \mathcal{S}_i \\ |y \cap z|=r}} (B_i)_{x, y}, \end{aligned}$$

where b_r is the number $|\{\alpha \in \Gamma \mid y^\alpha = y'\}|$ and b_r is independent of the choice of y and y' .

Finally, for $0 \leq s \leq i-1$, we have

$$0 = |\Gamma|(B_i)_{x_s, z} = \sum_{r=0}^{i-1} c_r \sum_{\substack{y \in \mathcal{S}_i \\ |y \cap z|=r}} (B_i)_{x_s, y} = \sum_{r=0}^{i-1} c_r T_{r,s}, \quad \text{where } c_r = b_r \sum_{\substack{y' \in \mathcal{S}_i \\ |y' \cap z|=r}} a_{y'}.$$

Then $c_r = 0$ for all $0 \leq r \leq i-1$ since T is invertible. This implies

$$(B_i)_{x, z} = \frac{1}{|\Gamma|} \sum_{r=0}^{i-1} c_r \sum_{\substack{y \in \mathcal{S}_i \\ |y \cap z|=r}} (B_i)_{x, y} = 0.$$

This is a contradiction. □

Theorem 1.7. Let P, Q be the first and second eigenmatrices of the Johnson scheme $J(v, k)$ and let $\{\mu_i\}_{i=0}^k, \{n_j\}_{j=0}^k$ be the multiplicities and the valencies of $J(v, k)$ respectively. Then for $0 \leq i, j \leq k$,

$$\begin{aligned} P_{i,j} &= \frac{n_j}{\mu_i} Q_{j,i} = E_j(i) = \sum_{l=0}^j (-1)^{j-l} \binom{k-l}{j-l} \binom{k-i}{l} \binom{v-k+l-i}{l}, \\ \mu_i &= \binom{v}{i} - \binom{v}{i-1}, \quad n_j = \binom{k}{j} \binom{v-k}{j}. \end{aligned}$$

Here $E_j(x)$ is a so-called Eberlein polynomial and it has degree $2j$ in the indeterminate x .

Proof of Theorem 1.7. As in Lemma 1.6, set $C_i := B_i B_i^T$ ($0 \leq i \leq k$) and take primitive idempotents in order of Lemma 1.6 (iii). We check that for any $0 \leq r \leq k$, we have

$$C_r = \sum_{i=0}^r \binom{k-i}{r-i} \binom{v-r-i}{k-r} E_i, \quad (\star)$$

Let the coefficient of E_i be $\alpha_{r,i}$ and show that $\alpha_{r,i}$ has the form above. For $0 \leq s \leq r \leq k$,

$$\begin{aligned} C_r C_s &= \sum_{j=0}^s \alpha_{r,j} \alpha_{s,j} E_j \\ &= \sum_{j=0}^s (\alpha_{r,j} - \alpha_{r,s}) \alpha_{s,j} E_j + \sum_{j=0}^s \alpha_{r,s} \alpha_{s,j} E_j \\ &= \sum_{j=0}^{s-1} (\alpha_{r,j} - \alpha_{r,s}) \alpha_{s,j} E_j + \alpha_{r,s} C_s. \end{aligned}$$

Comparing the coefficient of C_s to the formula in Lemma 1.6 (ii), we have

$$\alpha_{r,s} = \binom{k-s}{r-s} \binom{k-s}{s-s} \binom{v-r-s}{v-k-s} = \binom{k-s}{r-s} \binom{v-r-s}{v-k-s}.$$

This implies (\star) . Finally, we have

$$\begin{aligned} &\sum_{i=0}^k \left(\sum_{l=0}^{k-i} (-1)^{j-l} \binom{k-l}{j-l} \binom{k-i}{l} \binom{v-k+l-i}{l} \right) E_i \\ &= \sum_{l=0}^k (-1)^{j-l} \binom{k-l}{j-l} \left(\sum_{i=0}^{k-l} \binom{k-i}{(k-l)-i} \binom{v-(k-l)-i}{k-(k-l)} E_i \right) \\ &= \sum_{l=0}^k (-1)^{j-l} \binom{k-l}{j-l} C_{k-l} && \text{by } (\star) \\ &= \sum_{r=0}^k (-1)^{j-k+r} \binom{r}{j-k+r} C_r && r = k-l \\ &= \sum_{r=0}^k (-1)^{j-k+r} \binom{r}{j-k+r} \left(\sum_{l=r}^k \binom{l}{r} A_{k-l} \right) && \text{by Lemma 1.6 (i)} \\ &= \sum_{l=0}^k \sum_{r=0}^l (-1)^{j-k+r} \binom{r}{k-j} \binom{l}{r} A_{k-l} \\ &= \sum_{l=0}^k \sum_{r=0}^l (-1)^{j-k+r} \binom{l}{k-j} \binom{l-k+j}{r-k+j} A_{k-l} \\ &= \sum_{u=0}^k \sum_{r=0}^{k-u} (-1)^{j-k+r} \binom{k-u}{j-u} \binom{j-u}{r-k+j} A_u && u = k-l \end{aligned}$$

$$\begin{aligned}
&= A_j + \sum_{u=0}^{j-1} \sum_{r=0}^{k-u} (-1)^{j-k+r} \binom{k-u}{j-u} \binom{j-u}{r-k+j} A_u \\
&= A_j + \sum_{u=0}^{j-1} (1 + (-1))^{j-u} \binom{k-u}{j-u} A_u \\
&= A_j.
\end{aligned}$$

The valency of $J(v, k)$ is

$$n_j = |\{y \in X \mid d_J(x, y) = j\}| = \binom{k}{j} \binom{v-k}{j}.$$

For the multiplicities of $J(v, k)$, the equation $(*)$ says that $\text{rank}(C_r) = \sum_{i=0}^r \mu_i$. Then

$$\mu_i = \text{rank}(C_i) - \text{rank}(C_{i-1}) = \binom{v}{i} - \binom{v}{i-1} \quad \text{by Lemma 1.6 (iv).}$$

□

By using transformation formulas related to hypergeometric functions, we get other expressions of the eigenmatrices of the Johnson scheme. For a nonnegative integer i , define the Pochhammer symbol by

$$[a]_i = \begin{cases} a(a+1)\cdots(a+i-1) & \text{if } i \neq 0, \\ 1 & \text{if } i = 0. \end{cases} \quad (1.8)$$

For a nonnegative integer n and parameters a, b, c, d such that $[c]_n [d]_n \neq 0$, define the hypergeometric series by

$${}_3F_2 \left(\begin{matrix} -n, a, b \\ c, d \end{matrix}; x \right) = \sum_{i=0}^n \frac{[-n]_i [a]_i [b]_i}{[c]_i [d]_i i!} x^i. \quad (1.9)$$

Theorem 1.8. *For the Johnson scheme, every entry of the second eigenmatrix can be expressed by the Hahn polynomial $H_i(j)$:*

$$Q_{j,i} = H_i(j) = \mu_i {}_3F_2 \left(\begin{matrix} -j, -i, -v+i-1 \\ -k, -v+k \end{matrix}; 1 \right),$$

where $\{\mu_i\}_{i=0}^k$ denote the multiplicities of $J(v, k)$. In particular, $Q_{j,i}$ is a polynomial of degree i in the indeterminate j , since the $H_i(j)$ has degree i in the indeterminate j .

Before the proof, we prepare some lemmas.

Lemma 1.9. *For a positive integer i and an integer a , we have*

$$[a]_i = \begin{cases} \frac{(a+i-1)!}{(a-1)!} & \text{if } 0 < a, \\ (-1)^i \frac{(-a)!}{(-a-i)!} & \text{if } i \leq -a, \\ 0 & \text{if } 0 \leq -a < i. \end{cases}$$

Proof. It is clear from the definition. □

Lemma 1.10. (i) *For nonnegative integers i, j, k, v with $i, j \leq k$ and $2k \leq v$, we have*

$$\begin{aligned}
&\sum_{l=0}^j (-1)^{j-l} \binom{k-l}{j-l} \binom{k-i}{l} \binom{v-k+l-i}{l} \\
&= (-1)^j \binom{k}{j} {}_3F_2 \left(\begin{matrix} -j, -k+i, v-k-i+1 \\ -k, 1 \end{matrix}; 1 \right).
\end{aligned}$$

(ii) For nonnegative integers i, j, k, v with $i, j \leq k$ and $2k \leq v$, we have

$$(-1)^j \frac{[1+j]_{k-i}}{[1]_{k-i}} \frac{[v-k-j+1]_j}{[-k+i-j]_j} = \binom{v-k}{j}.$$

Proof. (i) We have

$$\begin{aligned} & {}_3F_2 \left(\begin{matrix} -j, -k+i, v-k-i+1 \\ -k, 1 \end{matrix}; 1 \right) \\ &= \sum_{l=0}^j \frac{[-j]_l [-k+i]_l [v-k-i+1]_l}{[-k]_l [1]_l l!} \\ &= \sum_{l=0}^{\min\{j,k-i\}} \frac{[-j]_l [-k+i]_l [v-k-i+1]_l}{[-k]_l [1]_l l!} \\ &= \sum_{l=0}^{\min\{j,k-i\}} \frac{(-1)^l \frac{j!}{(j-l)!} \cdot (-1)^l \frac{(k-i)!}{(k-i-l)!} \cdot \frac{(v-k-i+l)!}{(v-k-i)!}}{(-1)^l \frac{k!}{(k-l)!} \cdot l! \cdot l!} \quad \text{by Lemma 1.9} \\ &= \sum_{l=0}^{\min\{j,k-i\}} (-1)^l \frac{j!}{k!} \cdot \frac{(k-l)!}{(j-l)!} \cdot \frac{(k-i)!}{l!(k-i-l)!} \cdot \frac{(v-k+l-i)!}{l!(v-k-i)!} \\ &= \sum_{l=0}^{\min\{j,k-i\}} (-1)^l \frac{j!(k-j)!}{k!} \cdot \frac{(k-l)!}{(j-l)!(k-j)!} \cdot \frac{(k-i)!}{l!(k-i-l)!} \cdot \frac{(v-k+l-i)!}{l!(v-k-i)!} \\ &= \sum_{l=0}^{\min\{j,k-i\}} (-1)^l \binom{k}{j}^{-1} \binom{k-l}{j-l} \binom{k-i}{l} \binom{v-k+l-i}{l} \\ &= \sum_{l=0}^j (-1)^l \binom{k}{j}^{-1} \binom{k-l}{j-l} \binom{k-i}{l} \binom{v-k+l-i}{l}. \end{aligned}$$

(ii) We have

$$\begin{aligned} & (-1)^j \frac{[1+j]_{k-i}}{[1]_{k-i}} \frac{[v-k-j+1]_j}{[-k+i-j]_j} = (-1)^j \frac{\frac{(k-i+j)!}{j!}}{\frac{(k-i)!}{(k-i)!}} \cdot \frac{\frac{(v-k)!}{(v-k-j)!}}{\frac{(-1)^j \frac{(k-i+j)!}{(k-i)!}}{(-1)^j \frac{(k-i)!}{(k-i)!}}} \quad \text{by Lemma 1.9} \\ &= \frac{(v-k)!}{j!(v-k-j)!} = \binom{v-k}{j}. \end{aligned}$$

□

We need the following transformation formula between two hypergeometric series:

Theorem 1.11 (Andrews–Askey–Roy [1, Corollary 3.3.4]). *For any nonnegative integer n and any parameters a, b, c, d with $[c]_n \neq 0$, $[d]_n \neq 0$ and $[d-b]_n \neq 0$, we have*

$${}_3F_2 \left(\begin{matrix} -n, a, b \\ c, d \end{matrix}; 1 \right) = \frac{[d-b]_n}{[d]_n} {}_3F_2 \left(\begin{matrix} -n, c-a, b \\ c, 1+b-d-n \end{matrix}; 1 \right).$$

Proof of Theorem 1.8. For the (j, i) -entry of the second eigenmatrix, we have

$$\begin{aligned} Q_{j,i} &= \frac{\mu_i}{\binom{k}{j} \binom{v-k}{j}} \sum_{l=0}^j (-1)^{j-l} \binom{k-l}{j-l} \binom{k-i}{l} \binom{v-k+l-i}{l} \\ &= \frac{\mu_i}{\binom{k}{j} \binom{v-k}{j}} (-1)^j \binom{k}{j} {}_3F_2 \left(\begin{matrix} -j, -k+i, v-k-i+1 \\ -k, 1 \end{matrix}; 1 \right) \quad \text{by Lemma 1.10 (i)} \\ &= \frac{\mu_i}{\binom{v-k}{j}} (-1)^j {}_3F_2 \left(\begin{matrix} -k+i, v-k-i+1, -j \\ -k, 1 \end{matrix}; 1 \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{\mu_i}{\binom{v-k}{j}} (-1)^j \frac{[1+j]_{k-i}}{[1]_{k-i}} {}_3F_2 \left(\begin{matrix} -k+i, -v+i-1, -j \\ -k, -k+i-j \end{matrix}; 1 \right) \quad \text{by Theorem 1.11} \\
&= \frac{\mu_i}{\binom{v-k}{j}} (-1)^j \frac{[1+j]_{k-i}}{[1]_{k-i}} {}_3F_2 \left(\begin{matrix} -j, -k+i, -v+i-1 \\ -k, -k+i-j \end{matrix}; 1 \right) \\
&= \frac{\mu_i}{\binom{v-k}{j}} (-1)^j \frac{[1+j]_{k-i}}{[1]_{k-i}} \\
&\quad \times \frac{[v-k-j+1]_j}{[-k+i-j]_j} {}_3F_2 \left(\begin{matrix} -j, -i, -v+i-1 \\ -k, -v+k \end{matrix}; 1 \right) \quad \text{by Theorem 1.11} \\
&= \mu_i {}_3F_2 \left(\begin{matrix} -j, -i, -v+i-1 \\ -k, -v+k \end{matrix}; 1 \right) \quad \text{by Lemma 1.10 (ii).}
\end{aligned}$$

□

We will finish this section with an important property of Johnson schemes. In later chapter, we will use this property, which is usually called the Q -polynomial property.

Theorem 1.12. *The Krein parameters $\{q_{i,j}^h\}_{i,j,h=0}^k$ of the Johnson scheme $J(v,k)$ with respect to the ordering in Theorem 1.7 has the property that $q_{i,j}^h = 0$ if $i+j > h$ and $q_{i,j}^h \neq 0$ if $i+j = h$.*

Proof. Let \mathcal{Q} be the second eigenmatrix of the Johnson scheme $J(v,k)$, which is determined in Theorem 1.7. Since Johnson schemes are symmetric, we have the following relation for \mathcal{Q} and $\{q_{i,j}^h\}_{i,j,h=0}^k$ by Theorem 1.4:

$$\mathcal{Q}_{m,i} \mathcal{Q}_{m,j} = \sum_{h=0}^k q_{i,j}^h \mathcal{Q}_{m,h} \quad (0 \leq i, j, m \leq k).$$

Then Theorem 1.8 tells us that the left-hand side has degree $(i+j)$ in the indeterminate m , while each of the terms in the right-hand side has degree h . This implies the property. □

Chapter 2

Gelfand Pairs

An appropriate pair of a finite group G and its subgroup H forms a commutative association scheme. This pair is called a Gelfand pair, which is strongly related to spherical functions in group theory and symmetric spaces in differential geometry. In this chapter, we will discuss the relations between Gelfand pairs and commutative association schemes and bring them into general situations.

2.1 Equivalent Definitions of Gelfand Pairs

Gelfand pairs have several equivalent definitions including the definition above in terms of commutative association schemes. In this section, we begin with one typical definition and show some other equivalent definitions. Then we will see the relations between Gelfand pairs and commutative association schemes at the end of this section. This section is based on Bannai–Ito [2] and Terras [25].

For a subgroup H of a finite group G , let $L(H\backslash G/H)$ be the set of left and right H -invariant functions from G to the complex number field \mathbb{C} :

$$L(H\backslash G/H) = \{f : G \rightarrow \mathbb{C} \mid f(axb) = f(x) \text{ for all } a, b \in H\}. \quad (2.1)$$

This set $L(H\backslash G/H)$ is a \mathbb{C} -algebra under convolution. Here the convolution of two functions $f, g : G \rightarrow \mathbb{C}$ is defined by

$$(f * g)(x) = \sum_{y \in G} f(xy^{-1})g(y) = \sum_{y \in G} f(y)g(y^{-1}x). \quad (2.2)$$

Then the pair (G, H) is called a Gelfand pair if the algebra $L(H\backslash G/H)$ is commutative under convolution.

Let $L(G/H)$ be the set of right H -invariant functions from G to \mathbb{C} :

$$L(G/H) = \{f : G \rightarrow \mathbb{C} \mid f(xh) = f(x) \text{ for all } h \in H\}. \quad (2.3)$$

It can be seen that G acts on $L(G/H)$. Let $\lambda : G \rightarrow \mathrm{GL}(L(G/H))$ be the unitary representation of G associated to this action:

$$\lambda(g)f(x) = f(g^{-1}x). \quad (2.4)$$

Theorem 2.1. *Let $\mathrm{Hom}_G(L(G/H), L(G/H))$ be the set of all linear maps from $L(G/H)$ to $L(G/H)$ which commute with λ . Then $L(H\backslash G/H)$ is anti-isomorphic as algebra to $\mathrm{Hom}_G(L(G/H), L(G/H))$.*

Proof. For $f \in L(H\backslash G/H)$ and $g \in L(G/H)$, the convolution $g * f$ is a right H -invariant function since f is right H -invariant. Then, define the algebra anti-homomorphism φ from $L(H\backslash G/H)$ to $\mathrm{Hom}_G(L(G/H), L(G/H))$ by

$$[\varphi(f)](g) = g * f \quad \text{for all } f \in L(H\backslash G/H), g \in L(G/H).$$

On the other hand, the map φ' from $\mathrm{Hom}_G(L(G/H), L(G/H))$ to $L(H\backslash G/H)$ is defined by

$$\varphi'(F) = \frac{1}{|H|} F(\delta_H) \quad \text{for all } F \in \mathrm{Hom}_G(L(G/H), L(G/H)),$$

where $\delta_H \in L(G/H)$ is the characteristic function of H , i.e. $\delta_H(x)$ equals 1 if x is in H and 0 otherwise. Note that $F(\delta_H)$ is in $L(H \setminus G/H)$, that is the map φ' is well-defined, because for $h \in H, x \in G$ we have

$$[F(\delta_H)](hx) = [\lambda(h^{-1})F(\delta_H)](x) = [F(\lambda(h^{-1})\delta_H)](x) = [F(\delta_H)](x).$$

The map φ' is inverse to φ because

$$\begin{aligned} [\varphi'\varphi(f)](x) &= \frac{1}{|H|}(\delta_H * f)(x) = \frac{1}{|H|} \sum_{h \in H} f(h^{-1}x) = f(x), \\ [\varphi\varphi'(F)](g) &= \frac{1}{|H|}(g * F(\delta_H)) = \frac{1}{|H|}F(g * \delta_H) = F(g). \end{aligned}$$

Here $f \in L(H \setminus G/H)$, $g \in L(G/H)$, $F \in \text{Hom}_G(L(G/H), L(G/H))$ and $x \in G$. Thus φ is bijective and, in particular, the desired anti-isomorphism. \square

The next theorem characterizes Gelfand pairs.

Theorem 2.2. *(G, H) is a Gelfand pair if and only if $L(G/H)$ can be decomposed into a direct sum of mutually orthogonal irreducible G -spaces. This is also equivalent to the condition that the induced representation of the trivial representation in H is multiplicity-free.*

Proof. Suppose we have a direct sum decomposition $L(G/H) = \bigoplus_{i=0}^r V_i$, where the V_i are irreducible G -spaces. This decomposition implies the decomposition of a linear map $F \in \text{Hom}_G(L(G/H), L(G/H))$ as $F = \bigoplus_{i,j} F_{i,j}$, where $F_{i,j}$ is a map from V_i to V_j . Since the $F_{i,j}$ commute with the G -action and the V_i are irreducible G -spaces, each $F_{i,j}$ is a scalar map if $V_i \simeq V_j$ and the zero map otherwise (by Schur's lemma). Thus $\text{Hom}_G(L(G/H), L(G/H))$ is isomorphic to a direct sum of full matrix algebras, and it is commutative if and only if the V_i are not equivalent with each other. \square

Recall a basic fact about association schemes constructed from finite groups. This fact leads to the relation between Gelfand pairs and association schemes.

Theorem 2.3 (Brouwer–Haemers [4, Section 4.5]). *Let G be a finite group acting on the coset G/H , where H is a subgroup of G . Then G has a natural action on $(G/H) \times (G/H)$. If the induced representation of the trivial representation in H is multiplicity-free, the orbitals of the action form a commutative association scheme.*

Proof. This comes from the fact that $\text{Hom}_G(L(G/H), L(G/H))$ is the Bose–Mesner algebra of the association scheme. \square

Corollary. *Let H be a subgroup of a finite group G . (G, H) is a Gelfand pair if and only if the orbitals of the G -action on $(G/H) \times (G/H)$ form a commutative association scheme.*

2.2 Generalized Theory of Gelfand Pairs

Considering the H -bi-invariant function space $L(H \setminus G/H)$, we have seen a Gelfand pair forms a commutative association scheme. This section will focus on (H, K) -bi-invariant function space $L(H \setminus G/K)$ instead, where (G, H) and (G, K) are two Gelfand pairs.

First we observe two Gelfand pairs. Let H, K be two subgroups of a finite group G such that the induced representations of the trivial representations $1_H, 1_K$ are multiplicity-free. In other words, there exist irreducible decompositions:

$$L(G/H) = V_0 \perp V_1 \perp \cdots \perp V_m, \quad L(G/K) = W_0 \perp W_1 \perp \cdots \perp W_n. \quad (2.5)$$

Without loss of generality, we shall assume V_i and W_i are equivalent as G -space for $0 \leq i \leq d$ and not for any other pairs (V_i, W_j) . Let $L(H \setminus G/K)$ be the set of left H -invariant and right K -invariant functions from G to \mathbb{C} :

$$L(H \setminus G/K) = \{f : G \rightarrow \mathbb{C} \mid f(hxk) = f(x) \text{ for all } h \in H, k \in K\}. \quad (2.6)$$

In other words, $L(H \setminus G/K)$ is the set of left H -invariant functions in $L(G/K)$.

Theorem 2.4. For all $0 \leq i \leq n$, let $(W_i)^H$ be the set of left H -invariant functions in W_i . Then we have

$$\dim(W_i)^H = \begin{cases} 1 & \text{if } 0 \leq i \leq d, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the dimension of the function space $L(H \setminus G/K)$ is $d+1$.

Proof. By the Frobenius reciprocity law, we have

$$\dim(W_i)^H = \langle 1_H, W_i|_H \rangle_H = \langle \text{Ind}_H^G(1_H), W_i \rangle_G = \begin{cases} 1 & \text{if } 0 \leq i \leq d, \\ 0 & \text{otherwise.} \end{cases}$$

□

Consider the componentwise action of G on the Cartesian product of two cosets $(G/K) \times (G/H)$. Recall Theorem 2.3 which says that this action forms a commutative association scheme in the case $H = K$. We will generalize this method under some assumptions.

Let $\{x_0, x_1, \dots, x_d\}$ denote a set of complete representatives of the double cosets $H \setminus G/K$, where we assume $x_0 = e_G$, the identity element in the group G . Since the orbitals of $(x_i K, H)$ are different each other and cover the entire space, we have the orbit decomposition of the G -action as $(G/K) \times (G/H) = \sqcup_{i=0}^d R_i$, where $(x_i K, H) \in R_i$. Then define the adjacency matrix A_i of R_i by

$$(A_i)_{xK, yH} = \begin{cases} 1 & \text{if } (xK, yH) \in R_i, \\ 0 & \text{otherwise.} \end{cases} \quad (2.7)$$

The matrix A_i can be seen as a linear function from $L(G/H)$ to $L(G/K)$. Note that A_i commutes with the G -action because R_i is an orbital of the action. By considering the action on $(G/H) \times (G/K)$, we get that A_i^T is also commutative with the G -action, i.e. $A_i \in \text{Hom}_G(L(G/H), L(G/K))$ and $A_i^T \in \text{Hom}_G(L(G/K), L(G/H))$.

Define another function $E_i : L(G/H) \rightarrow L(G/K)$ ($0 \leq i \leq d$) by

$$E_i|_{V_j} = \begin{cases} \pi_i & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases} \quad (2.8)$$

where $\pi_i : V_i \rightarrow W_i$ is an isomorphism of G -representation. Similarly, $E'_i : L(G/K) \rightarrow L(G/H)$ ($0 \leq i \leq d$) is defined by

$$E'_i|_{W_j} = \begin{cases} \pi_i^{-1} & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \quad (2.9)$$

Note that the isomorphisms π_i are uniquely determined up to scalar multiple by Schur's lemma. Similarly to A_i , the diagonal operators E_i are also in $\text{Hom}_G(L(G/H), L(G/K))$ and E'_i in $\text{Hom}_G(L(G/K), L(G/H))$.

In the case that $H = K$, the matrix A_i and E_i are the adjacency matrices and the primitive idempotents up to scalar multiple of the association scheme obtained by the G -action. As the case $H = K$, we define the two matrices P and Q by

$$A_j^T = \sum_{i=0}^d P_{i,j} E'_i, \quad E_i = \frac{1}{|R_0|} \sum_{j=0}^d Q_{j,i} A_j. \quad (2.10)$$

The next theorem is the relation between two matrices P and Q , which is a generalization of Theorem 1.3.

Theorem 2.5. For any integers $0 \leq i, j \leq d$, we have

$$\frac{|R_j|}{|R_0|} Q_{j,i} = P_{i,j} \dim V_i.$$

Proof. We consider the trace of $A_j^T E_i$ in two ways,

$$\begin{aligned}\mathrm{Tr}(A_j^T E_i) &= \frac{1}{|R_0|} \sum_{k=0}^d Q_{k,i} \mathrm{Tr}(A_j^T A_k) = \frac{1}{|R_0|} Q_{j,i} \mathrm{Tr}(A_j^T A_j) = \frac{1}{|R_0|} Q_{j,i} |R_j|, \\ \mathrm{Tr}(A_j^T E_i) &= \sum_{k=0}^d P_{k,j} \mathrm{Tr}(E'_k E_i) = P_{i,j} \mathrm{Tr}(E'_i E_i) = P_{i,j} \dim V_i.\end{aligned}$$

□

All entries of the matrix Q are determined by the matrix P by Theorem 2.5. At last, we will construct a formula to determine entries of the matrix P . Considering the following isomorphism, as a generalization of Theorem 2.1, we omit the proof because this would be exactly the same as that of Theorem 2.1.

Theorem 2.6. Define $\varphi : \mathrm{Hom}_G(L(G/H), L(G/K)) \rightarrow L(H \backslash G/K)$ by

$$\varphi(F) = \frac{1}{|H|} F(\delta_H).$$

Then φ is an isomorphism as vector space.

From this isomorphism, we express the matrix P using the terms of $L(H \backslash G/K)$ called the spherical functions ω_i . For each $0 \leq i \leq d$, there exists a non-zero function ω_i in W_i such that $\lambda(h)\omega_i = \omega_i$ for all $h \in H$, and it is unique up to scalar multiple by Theorem 2.4. From now on, we assume the value $\omega_i(e_G)$ is nonzero for all i and normalize the value to be 1. Then such a function ω_i is called the spherical function in W_i . By definition, the spherical functions $\omega_0, \omega_1, \dots, \omega_d$ form an orthogonal basis of $L(H \backslash G/K)$.

Before going to the next theorem, we describe $\varphi(A_i)$ and $\varphi(E_i)$ by direct calculation.

$$\varphi(A_i) = \frac{1}{|H|} A_i(\delta_H) = \frac{1}{|H|} \sum_{\substack{yK \in G/K \\ (yK, H) \in R_i}} \delta_{yK} = \frac{1}{|H|} \sum_{yK \in Hx_i K} \delta_{yK} = \frac{1}{|H|} \delta_{Hx_i K}. \quad (2.11)$$

Since $\varphi(E_j) \in W_j^H$, there exists $\lambda_j \in \mathbb{C}$ such that $\varphi(E_j) = \lambda_j \omega_j$. By comparing the value at e_G , the identity element of G , we have

$$\lambda_j = \varphi(E_j)(e_G) = \frac{1}{|R_0|} \sum_{i=0}^d Q_{i,j} \varphi(A_i)(e_G) = \frac{1}{|R_0||H|} \sum_{i=0}^d Q_{i,j} \delta_{Hx_i K}(e_G) = \frac{1}{|R_0||H|} Q_{0,j}.$$

Thus we have

$$\varphi(E_j) = \frac{Q_{0,j}}{|R_0||H|} \omega_j \quad (2.12)$$

Theorem 2.7. For any integers $0 \leq i, j \leq d$, we have

$$P_{j,i} = \frac{|R_i| Q_{0,j}}{|R_0| \dim V_j} \omega_j(x_i) = \frac{|R_i|}{|R_0|} P_{j,0} \omega_j(x_i).$$

Proof.

$$\begin{aligned}P_{j,i} &= \frac{|R_i|}{|R_0| \dim V_j} Q_{0,j} && \text{by Theorem 2.5} \\ &= \frac{|R_i|}{|R_0| \dim V_j} \left(\sum_{k=0}^d Q_{k,j} \delta_{Hx_k K}(x_i) \right) \\ &= \frac{|R_i|}{|R_0| \dim V_j} \left(\sum_{k=0}^d Q_{k,j} |H| \varphi(A_k)(x_i) \right) && \text{by (2.11)} \\ &= \frac{|R_i|}{|R_0| \dim V_j} |H| \varphi \left(\sum_{k=0}^d Q_{k,j} A_k \right) (x_i)\end{aligned}$$

$$\begin{aligned}
&= \frac{|R_i|}{|R_0| \dim V_j} |H| \varphi(|R_0| E_j)(x_i) && \text{by (2.10)} \\
&= \frac{|R_i|}{\dim V_j} |H| \varphi(E_j)(x_i) \\
&= \frac{|R_i|}{|R_0| \dim V_j} Q_{0,j} \omega_j(x_i) && \text{by (2.12)} \\
&= \frac{|R_i|}{|R_0|} P_{j,0} \omega_j(x_i) && \text{by Theorem 2.5.}
\end{aligned}$$

□

To summarize, this section has shown that the G -action on $(G/K) \times (G/H)$ does not form an association scheme in general, but brings the similar structure under the assumption that there exist spherical functions in $L(H \backslash G/K)$. To construct the structure, we defined the adjacency matrices A_i and the orthogonal projections E_i which correspond to the adjacency matrices and the primitive idempotents in a structure of association schemes. We also defined two matrices P and Q which correspond to the first and second eigenmatrices. In addition, we have shown the relation between the two matrices P and Q (Theorem 2.5) and that the entries of the matrix P are expressed by the values of the spherical functions and some parameters (Theorem 2.7). In Chapter 4, we will use this structure and see its utility.

Chapter 3

Delsarte's Linear Programming Method

This chapter considers subsets of a symmetric association scheme with d classes satisfying some restrictions with respect to its relations. We are interested in the following question: what is the maximal/minimal size of such a subset? We introduce linear programming methods to approach this question, which is constructed by Delsarte [6]. In the latter half of this chapter, we present one new theorem in design theory as its applications.

3.1 Delsarte's Linear Programming Bound

Let $(X, \{R_i\}_{i=0}^d)$ be a symmetric association scheme and let $\{A_i\}_{i=0}^d$ be its adjacency matrices. For a non-empty subset $Y \subset X$, the inner distribution of Y with respect to $\{R_i\}_{i=0}^d$ is the row vector $\mathbf{a} = (a_0 \ a_1 \ \dots \ a_d)$ whose entries are defined by

$$a_i = \frac{|(Y \times Y) \cap R_i|}{|Y|} = \frac{1}{|Y|} \chi^T A_i \chi,$$

where χ is the characteristic vector of Y . It is clear that $a_0 = 1$ and $\sum_{i=0}^d a_i = |Y|$.

Theorem 3.1 (Delsarte [6]). *The inner distribution of a non-empty subset of a symmetric association scheme satisfies $\mathbf{a}Q \geq 0$, where Q is the second eigenmatrix of the association scheme.*

Proof. For any integer $0 \leq j \leq d$, check if $(\mathbf{a}Q)_j$ is non-negative. Let $\{E_j\}_{j=0}^d$ be the primitive idempotents of the association scheme. We have

$$|Y|(\mathbf{a}Q)_j = |Y| \left(\sum_{i=0}^d a_i Q_{i,j} \right) = \chi^T \left(\sum_{i=0}^d Q_{i,j} A_i \right) \chi = |X| \chi^T E_j \chi \geq 0,$$

since E_j is positive semidefinite (idempotent and symmetric matrix). \square

For a remark, the 0-th entry of the row vector $\mathbf{a}Q$ is equal to $|Y|$ since the second eigenmatrix Q has all-one in the 0-th column. Delsarte constructed the linear programming bounds by using this inequality.

Suppose the inner distribution \mathbf{a} of Y has constraint linear conditions. For example, a condition would be $a_1 - 2a_2 = 0$. We consider a_0, a_1, \dots, a_d as variables and define

$$a^* = \max \sum_{i=0}^d a_i, \quad \text{subject to } \begin{cases} a_0 = 1, \\ a_i \geq 0 & \text{for } 1 \leq i \leq d, \\ (\mathbf{a}Q)_j \geq 0 & \text{for } 1 \leq j \leq d, \\ \text{linear conditions on } \mathbf{a}. \end{cases} \quad (3.1)$$

Then clearly a^* is an upper bound for the cardinality of a subset Y . By replacing maximize to minimize in the linear programming problem, a^* will become a lower bound. This is called Delsarte's linear programming method.

On applying into some specific association schemes, the linear programming method implies some useful results in different fields, such as upper bounds for the number of code-words with a given length and minimum distance and another proof of the Erdős–Ko–Rado theorem. See Brouwer–Haemers [4] for more examples. In the next section, we focus on one application to design theory and introduce a new theorem obtained from Delsarte’s linear programming method.

3.2 Applications to Design Theory

Let X be a set of size v and let X_k be the set consisting of all k -subsets of X , where $2k \leq v$. Here a k -subset always means a subset of size k . In design theory, a t -(v, k, λ) design (t -design) is defined to be a subset \mathcal{F} of X_k satisfying the following restriction: each t -subset of X is contained in precisely λ elements in \mathcal{F} . One of the fundamental problems in design theory is about the existence of a t -(v, k, λ) design with given parameters t, v, k, λ .

Before applying Delsarte’s linear programming method, we briefly review some known facts about t -designs (see [5]). Each element of a t -design \mathcal{F} is called a block and for any s -subset S of X , with $0 \leq s \leq t$, the number of blocks containing S is $\lambda_s = \lambda \binom{v-s}{t-s} / \binom{k-s}{t-s}$. In particular λ_s is an integer and \mathcal{F} is an s -(v, k, λ_s) design for all $0 \leq s \leq t$. The number of blocks $|\mathcal{F}|$ is λ_0 . In addition, theorems below are well-known facts about t -(v, k, λ) designs. See [5].

Theorem 3.2. *Let \mathcal{F} be a t -(v, k, λ) design. For any $0 \leq i \leq t$, we have*

$$\lambda_i = \frac{|\mathcal{F}|}{|X_k|} \binom{v-i}{k-i}.$$

Theorem 3.3. *For given integers $0 < t < k < v$, let λ_{\min} be the smallest positive integer λ such that each $\lambda_s = \lambda \binom{v-s}{t-s} / \binom{k-s}{t-s}$ for $0 \leq s \leq t$ is an integer. If a t -(v, k, λ) design exists, λ_{\min} divides λ .*

Theorem 3.4 (Fisher type bounds). *Let \mathcal{F} be a t -(v, k, λ) design. Then the number of blocks satisfies the condition: $|\mathcal{F}| \geq \binom{v}{\lfloor t/2 \rfloor}$.*

As we see above, λ_0 is nothing but the number of blocks. In particular, the number of blocks of a t -(v, k, λ) design is determined by its parameters t, v, k, λ . Therefore, we want to obtain necessary conditions on the number of blocks, such as Fisher type bounds. By applying Delsarte’s linear programming method to the Johnson scheme, which is the association scheme based on X_k , we have lower bounds for the number of blocks. We will see this in detail. For the first step, we need to characterize t -designs in terms of inner distributions.

Theorem 3.5. *Let $\mathbf{a} = (a_0 \ a_1 \ \dots \ a_k)$ be the inner distribution of a subset \mathcal{F} of X_k with respect to the Johnson scheme $J(v, k)$ and let Q be the second eigenmatrix of $J(v, k)$. Then the following are equivalent:*

- (i) \mathcal{F} is a t -(v, k, λ) design for some λ .
- (ii) $(\mathbf{a}Q)_j = 0$ for any integer $1 \leq j \leq t$.

Proof. Let $(X_k, \{R_i\}_{i=0}^k)$ be the Johnson scheme $J(v, k)$. We add three statements and show that five statements (i) – (v) are equivalent to each other.

- (iii) For any $z \in X_i$ ($0 \leq i \leq t$),

$$|\{y \in \mathcal{F} \mid z \subset y\}| = \frac{|\mathcal{F}|}{|X_k|} \binom{v-i}{k-i}.$$

- (iv) For any $0 \leq i \leq t$, $B_i^T \chi_{\mathcal{F}} = \frac{|\mathcal{F}|}{|X_k|} B_i^T \chi_{X_k}$, where the B_i are the incidence matrices defined in (1.7). Here χ_S ($S \subset X_k$) denotes the characteristic vector of S .
- (v) For any $0 \leq i \leq t$, $E_i \chi_{\mathcal{F}} = \frac{|\mathcal{F}|}{|X_k|} E_i \chi_{X_k}$, where the E_i are the primitive idempotents of $J(v, k)$.

(i) \Rightarrow (iii). By Theorem 3.2, for any i -subset z of X ($0 \leq i \leq t$) we have

$$|\{y \in \mathcal{F} \mid z \subset y\}| = \lambda_i = \frac{|\mathcal{F}|}{|X_k|} \binom{v-i}{k-i}.$$

(iii) \Rightarrow (i). For any t -subset z of X , the number $|\{y \in \mathcal{F} \mid z \subset y\}|$ is independent of the choice of z . This is the definition of a t -design.

(iii) \Leftrightarrow (iv). For any i -subset z of X ($0 \leq i \leq t$), we have

$$(B_i^T \chi_{\mathcal{F}})(z) = |\{y \in \mathcal{F} \mid z \subset y\}|,$$

$$\frac{|\mathcal{F}|}{|X_k|} (B_i^T \chi_{X_k})(z) = \frac{|\mathcal{F}|}{|X_k|} |\{x \in X_k \mid z \subset x\}| = \frac{|\mathcal{F}|}{|X_k|} \binom{v-i}{k-i}.$$

(iv) \Leftrightarrow (v). This is the consequence of Lemma 1.6 (iii), which says that $B_i B_i^T$ is a linear combination of E_0, E_1, \dots, E_i and conversely E_i is a linear combination of $B_0 B_0^T, B_1 B_1^T, \dots, B_i B_i^T$.

(v) \Leftrightarrow (ii). For $1 \leq j \leq t$, we have

$$(\mathbf{a}Q)_j = \sum_{i=0}^k a_i Q_{i,j} = \frac{|X_k|}{|\mathcal{F}|} \chi_{\mathcal{F}}^T E_j \chi_{\mathcal{F}}, \quad \text{and} \quad \chi_{\mathcal{F}}^T E_j \chi_{X_k} = 0.$$

□

Theorem 3.6. Let Q be the second eigenmatrix of the Johnson scheme $J(v, k)$. Define

$$a_* = \min \sum_{i=0}^k a_i, \quad \text{subject to} \quad \begin{cases} a_0 = 1, \\ a_i \geq 0 & \text{for } 1 \leq i \leq k, \\ (\mathbf{a}Q)_j = 0 & \text{for } 1 \leq j \leq t, \\ (\mathbf{a}Q)_j \geq 0 & \text{for } t+1 \leq j \leq k. \end{cases} \quad (3.2)$$

Then a_* is a lower bound for the number of blocks of any t - (v, k, λ) designs.

Proof. This is a simple consequence of Theorem 3.5 and Delsarte's linear programming problem. □

If we add the restriction $a_i = 0$ for suitable i in Theorem 3.6, we obtain lower bounds for t -designs with restricted block intersections, such as quasi-symmetric designs. By this method, we can prove the nonexistence of a quasi-symmetric 2-(28, 7, 16) design with block intersections $\{1, 3\}$ and a quasi-symmetric 2-(29, 7, 12) design with block intersections $\{1, 3\}$. See Brouwer–Haemers [4].

Common problems of linear programming method are in difficulty of computing with extremely large parameters while Fisher type bounds depend only on parameters. We set up a new bound for the number of blocks of a t -design which does not require solving any linear programming problems. The main tool which we used is the duality theorem in linear programming.

Theorem 3.7 (Duality theorem, [16]). Let A be an $m \times n$ matrix. For column vectors \mathbf{b}, \mathbf{y} of size m and for column vectors \mathbf{c}, \mathbf{x} of size n , define

$$\begin{aligned} \text{Primary problem :} & \quad \text{maximize} \quad z = \mathbf{c}^T \mathbf{x}, \quad \text{subject to} \quad A \mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geq 0. \\ \text{Dual problem :} & \quad \text{minimize} \quad w = \mathbf{b}^T \mathbf{y}, \quad \text{subject to} \quad A^T \mathbf{y} \geq \mathbf{c}. \end{aligned}$$

Here we consider \mathbf{x}, \mathbf{y} as the variables. Then the optimal values z, w are equal if there exist solutions of both primary and dual problems.

On applying this duality theorem, Theorem 3.6 can be restated in terms of maximization as follows:

Theorem 3.8. Let Q be the second eigenmatrix of the Johnson scheme $J(v, k)$. Define

$$b_* = \max \sum_{j=0}^k Q_{0,j} b_j, \quad \text{subject to } \begin{cases} b_0 = 1, \\ b_j \leq 0 & t+1 \leq j \leq k, \\ (\mathbf{b}Q^T)_i \geq 0 & 1 \leq i \leq k. \end{cases} \quad (3.3)$$

Then b_* is a lower bound for the number of blocks of any t - (v, k, λ) designs.

The lower bound is the largest value of the objective function. In other words, any feasible solutions of (3.3), which are vectors satisfying all constraints, lead to lower bounds for the number of blocks of any t -designs.

Theorem 3.9. Let \mathcal{F} be a t - (v, k, λ) design and let Q be the second eigenmatrix of the Johnson scheme $J(v, k)$. Set $e = \lfloor t/2 \rfloor$. For any positive integer $m \leq t$, the number of blocks satisfies the inequality:

$$|\mathcal{F}| \geq \binom{v}{e} + \frac{Q_{0,m}}{\binom{v}{e}} \min \left\{ \frac{(\sum_{l=0}^e Q_{i,l})^2}{|Q_{i,m}|} \mid 1 \leq i \leq k, Q_{i,m} < 0 \right\}.$$

Before the proof, remark that this bound does not require solving any linear programming problems and that this is no less than the well-known Fisher type bound, which equals the first term, since the second term is non-negative.

Proof of Theorem 3.9. Define $\mu'_e = \sum_{i=0}^e \mu_i$, where the μ_i are the multiplicities of the Johnson scheme $J(v, k)$. Define a vector $\mathbf{x} = (x_0 \ x_1 \ \dots \ x_k)$ by

$$x_j = \frac{1}{\mu'_e} \sum_{l_1=0}^e \sum_{l_2=0}^e q_{l_1, l_2}^j,$$

where the q_{l_1, l_2}^j are the Krein parameters of $J(v, k)$. We show that \mathbf{x} is a feasible solution of (3.3) and construct a new feasible solution from \mathbf{x} .

The vector \mathbf{x} satisfies all conditions in (3.3) because

$$\begin{aligned} x_0 &= \frac{1}{\mu'_e} \sum_{l_1=0}^e \sum_{l_2=0}^e q_{l_1, l_2}^0 = \frac{1}{\mu'_e} \sum_{l_1=0}^e \mu_{l_1} = 1 && \text{by Theorem 1.4 (ii),} \\ x_{t+1} &= x_{t+2} = \dots = x_k = 0 && \text{by Theorem 1.12.} \end{aligned}$$

In addition, for $1 \leq i \leq k$,

$$\begin{aligned} (\mathbf{x}Q^T)_i &= \sum_{j=0}^k Q_{i,j} x_j = \frac{1}{\mu'_e} \sum_{l_1=0}^e \sum_{l_2=0}^e \left(\sum_{j=0}^k q_{l_1, l_2}^j Q_{i,j} \right) \\ &= \frac{1}{\mu'_e} \sum_{l_1=0}^e \sum_{l_2=0}^e Q_{i,l_1} Q_{i,l_2} && \text{by Theorem 1.4 (iii)} \\ &= \frac{1}{\mu'_e} \left(\sum_{l=0}^e Q_{i,l} \right)^2 \geq 0. \end{aligned}$$

Define a vector $\mathbf{y} = (y_0 \ y_1 \ \dots \ y_k)$ by changing only the m -th coordinate:

$$y_i = x_i \ (i \neq m), \quad y_m = x_m + \frac{1}{\mu'_e} \min \left\{ \frac{(\sum_{l=0}^e Q_{i,l})^2}{|Q_{i,m}|} \mid 1 \leq i \leq k, Q_{i,m} < 0 \right\}.$$

y_m is well-defined, i.e. the set is not empty because of Theorem 1.4 (iv). Two conditions $y_0 = 1$ and $y_{t+1} = y_{t+2} = \dots = y_k = 0$ are shown immediately from the fact about \mathbf{x} . Finally, for all $1 \leq i \leq k$ we have

$$(\mathbf{y}Q^T)_i = (\mathbf{x}Q^T)_i + Q_{i,m} \cdot \frac{1}{\mu'_e} \min \left\{ \frac{(\sum_{l=0}^e Q_{h,l})^2}{|Q_{h,m}|} \mid 1 \leq h \leq k, Q_{h,m} < 0 \right\}$$

$$\begin{aligned}
&= \frac{1}{\mu'_e} \left(\sum_{l=0}^e Q_{i,l} \right)^2 + Q_{i,m} \cdot \frac{1}{\mu'_e} \min \left\{ \frac{(\sum_{l=0}^e Q_{h,l})^2}{|Q_{h,m}|} \mid 1 \leq h \leq k, Q_{h,m} < 0 \right\} \\
&\geq \begin{cases} \frac{1}{\mu'_e} \left(\sum_{l=0}^e Q_{i,l} \right)^2 & \text{if } Q_{i,m} \geq 0 \\ \frac{1}{\mu'_e} \left(\sum_{l=0}^e Q_{i,l} \right)^2 + Q_{i,m} \cdot \frac{1}{\mu'_e} \frac{(\sum_{l=0}^e Q_{i,l})^2}{|Q_{i,m}|} & \text{if } Q_{i,m} < 0 \end{cases} \\
&\geq 0.
\end{aligned}$$

Thus \mathbf{y} is a feasible solution of (3.3) and the lower bound y_* for the number of blocks of any t - (v, k, λ) designs is

$$\begin{aligned}
y_* &= \sum_{j=0}^k Q_{0,j} y_j \\
&= \sum_{j=0}^k Q_{0,j} x_j + \frac{Q_{0,m}}{\mu'_e} \min \left\{ \frac{(\sum_{l=0}^e Q_{i,l})^2}{|Q_{i,m}|} \mid 1 \leq i \leq k, Q_{i,m} < 0 \right\} \\
&= \frac{1}{\mu'_e} \left(\sum_{l=0}^e Q_{0,l} \right)^2 + \frac{Q_{0,m}}{\mu'_e} \min \left\{ \frac{(\sum_{l=0}^e Q_{i,l})^2}{|Q_{i,m}|} \mid 1 \leq i \leq k, Q_{i,m} < 0 \right\} \\
&= \mu'_e + \frac{Q_{0,m}}{\mu'_e} \min \left\{ \frac{(\sum_{l=0}^e Q_{i,l})^2}{|Q_{i,m}|} \mid 1 \leq i \leq k, Q_{i,m} < 0 \right\} \quad \text{by Theorem 1.4 (i).}
\end{aligned}$$

This bound equals what we wanted to show because we have

$$\mu'_e = \sum_{i=0}^e \mu_i = \sum_{i=0}^e \left\{ \binom{v}{i} - \binom{v}{i-1} \right\} = \binom{v}{e}$$

by Theorem 1.7. \square

The vital point of this proof is the part of constructing \mathbf{y} from the feasible solution \mathbf{x} which is found by P. Delsarte. In the field of linear programming, this construction is in part of the simplex method which provides an optimal solution from one feasible solution.

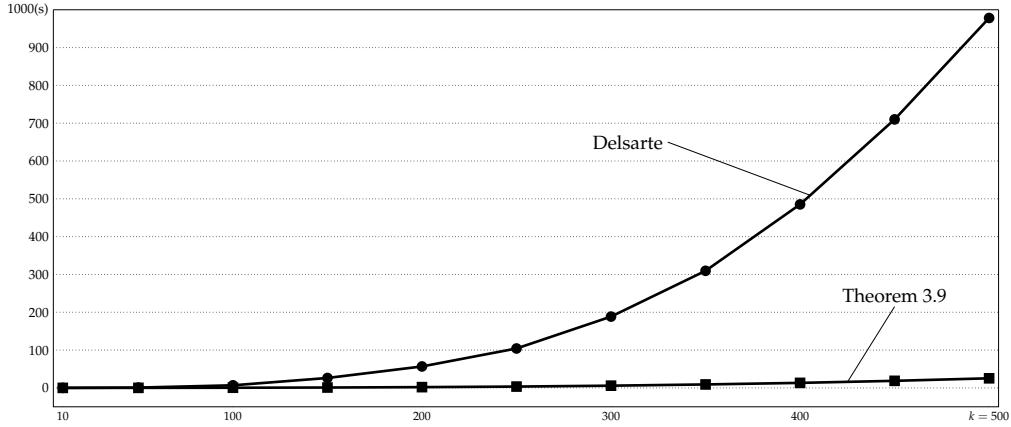
The main concern in design theory is whether a t - (v, k, λ) design with given parameters exists or not. From this viewpoint, the advantages of Theorem 3.9 also can be seen. Some parameters of t -designs which are ruled out by Theorem 3.9 but not by the Fisher type bound are listed below:

$$\begin{aligned}
(t, v, k, \lambda) &= (3, 22, 7, 1), (4, 12, 6, 2), (4, 17, 7, 2), (4, 23, 8, 2), (4, 38, 10, 2), \\
&\quad (4, 42, 16, 14), (4, 47, 11, 2), (4, 57, 12, 2), (6, 39, 18, 52), (6, 53, 17, 13).
\end{aligned}$$

See Appendix C.1 for the detailed results. (See also Appendix B.1.)

In addition, there is another advantage of Theorem 3.9 comparing with Delsarte's linear programming bounds. That is the computing time. For example, we calculate the bounds from both theorems for $t = 4$ and $v = 1000$ with the parameter $k = 10, 50, 100, 150, \dots, 500$. Figure 3.1 shows how faster new bounds are calculated.¹ For $k = 50$, only a few seconds of the CPU time were required in both cases. However, for $k = 500$, the time requirement of the Delsarte's bound is almost ten times greater than that of our new bound. This is because our new bounds do not require solving any linear programming problems.

¹By Mathematica 9.0 for Mac OS X x86 (64-bit) in Mac OS X 10.9 (1.6 GHz Intel Core i5).

Figure 3.1: Comparison between new and Delsarte's bounds for 4-(1000, k , λ) designs

In Theorem 3.9, the bound has a degree of freedom for the choice of an integer m . By taking special values for m , we can simplify the inequality.

Corollary. Let \mathcal{F} be a t -(v, k, λ) design and let Q be the second eigenmatrix of the Johnson scheme $J(v, k)$. Set $e = \lfloor t/2 \rfloor$. The number of blocks satisfies the inequality:

(i) If $v \geq k^2$,

$$|\mathcal{F}| \geq \binom{v}{e} + \frac{v-k}{\binom{v}{e}k} \left(\sum_{l=0}^e Q_{k,l} \right)^2.$$

(ii) If $v < k^2$ and $e = 1, 2$,

$$|\mathcal{F}| \geq \binom{v}{e} + \frac{v-1}{\binom{v}{e}} \min \left\{ \frac{1}{|Q_{i,1}|} \left(\sum_{l=0}^e Q_{i,l} \right)^2 \mid i = \lfloor \alpha_e \rfloor, \lceil \alpha_e \rceil, \quad i > \frac{k(v-k)}{v} \right\}.$$

(iii) If $t = 2, 3$ ($e = 1$),

$$|\mathcal{F}| \geq v + \frac{v-3}{2} \min \left\{ \frac{(1+Q_{i,2})^2}{|Q_{i,2}|} \mid i = \lfloor \alpha_1 \rfloor, \lceil \alpha_1 \rceil, \quad i < \alpha' \right\}.$$

Here α' is the maximum solution of the quadratic equation $Q_{x,2} = 0$ and α_e is the maximum solution of the algebraic equation $\sum_{l=0}^e Q_{x,l} = 0$.

Proof. By Theorem 1.7 and Theorem 1.8, we have

$$\begin{aligned} f_1(i) &:= Q_{i,1} = -\frac{v(v-1)}{k(v-k)}i + (v-1), \\ f_2(i) &:= Q_{i,2} \\ &= \frac{v(v-1)(v-2)(v-3)}{2k(k-1)(v-k)(v-k-1)}i^2 - \frac{v(v-1)(v-3)\{(2k-1)v-2k^2\}}{2k(k-1)(v-k)(v-k-1)}i + \frac{v(v-3)}{2}. \end{aligned}$$

We see $f_j(i)$ as polynomials of degree j in the indeterminate i .

α', β' denote two solutions of the quadratic equation $f_2(i) = 0$. ($\alpha' \geq \beta'$ as in the statement.) Similarly, α_1 denotes the solution of the equation $f_1(i) + 1 = 0$ and $\alpha_2 \geq \beta_2$ denote two solutions of the quadratic equation $f_2(i) + f_1(i) + 1 = 0$. Then we have

$$1 < \beta' < \alpha' < k, \tag{*}$$

$$\alpha_1 = \frac{k(v-k)}{v-1}, \tag{**}$$

$$\beta_2 < \frac{k(v-k)}{v} < \alpha_2 < k. \tag{***}$$

Let $m = 1$ and apply Theorem 3.9,

$$\begin{aligned} |\mathcal{F}| &\geq \binom{v}{e} + \frac{Q_{0,1}}{\binom{v}{e}} \min \left\{ \frac{(\sum_{l=0}^e Q_{i,l})^2}{|Q_{i,1}|} \mid 1 \leq i \leq k, Q_{i,1} < 0 \right\} \\ &= \binom{v}{e} + \frac{v-1}{\binom{v}{e}} \min \left\{ \frac{(\sum_{l=0}^e Q_{i,l})^2}{-Q_{i,1}} \mid \frac{k(v-k)}{v} < i \leq k \ (i \in \mathbb{Z}) \right\}. \end{aligned} \quad (\star)$$

Let $m = 2$ and apply Theorem 3.9,

$$\begin{aligned} |\mathcal{F}| &\geq \binom{v}{e} + \frac{Q_{0,2}}{\binom{v}{e}} \min \left\{ \frac{(\sum_{l=0}^e Q_{i,l})^2}{|Q_{i,2}|} \mid 1 \leq i \leq k, Q_{i,2} < 0 \right\} \\ &= \binom{v}{e} + \frac{v(v-3)}{2\binom{v}{e}} \min \left\{ \frac{(\sum_{l=0}^e Q_{i,l})^2}{|Q_{i,2}|} \mid 1 \leq i \leq k, \beta' < i < \alpha' \ (i \in \mathbb{Z}) \right\} \\ &= \binom{v}{e} + \frac{v(v-3)}{2\binom{v}{e}} \min \left\{ \frac{(\sum_{l=0}^e Q_{i,l})^2}{|Q_{i,2}|} \mid \beta' < i < \alpha' \ (i \in \mathbb{Z}) \right\} \quad \text{by } (*). \end{aligned} \quad (\star\star)$$

(i) If $v \geq k^2$, then the bound (\star) is

$$\begin{aligned} |\mathcal{F}| &\geq \binom{v}{e} + \frac{v-1}{\binom{v}{e}} \min \left\{ \frac{(\sum_{l=0}^e Q_{i,l})^2}{-Q_{i,1}} \mid k-1 \leq \frac{k(v-k)}{v} < i \leq k \ (i \in \mathbb{Z}) \right\} \\ &= \binom{v}{e} + \frac{v-1}{\binom{v}{e}} \frac{(\sum_{l=0}^e Q_{k,l})^2}{-Q_{k,1}} \\ &= \binom{v}{e} + \frac{v-1}{\binom{v}{e}} \frac{v-k}{k(v-1)} \left(\sum_{l=0}^e Q_{k,l} \right)^2 \\ &= \binom{v}{e} + \frac{v-k}{\binom{v}{e}k} \left(\sum_{l=0}^e Q_{k,l} \right)^2. \end{aligned}$$

(ii) If $v < k^2$ and $e = 1$, then the bound (\star) is

$$\begin{aligned} |\mathcal{F}| &\geq \binom{v}{1} + \frac{v-1}{\binom{v}{1}} \min \left\{ \frac{(\sum_{l=0}^1 Q_{i,l})^2}{-Q_{i,1}} \mid \frac{k(v-k)}{v} < i \leq k \ (i \in \mathbb{Z}) \right\} \\ &= \binom{v}{1} + \frac{v-1}{\binom{v}{1}} \min \left\{ \frac{(f_1(i)+1)^2}{-f_1(i)} \mid \frac{k(v-k)}{v} < i \leq k \ (i \in \mathbb{Z}) \right\}. \end{aligned}$$

For $-(f_1+1)^2/f_1$, we have

$$\frac{d}{di} \left(\frac{(f_1+1)^2}{-f_1} \right) = \frac{d}{di} \left(-f_1 - 2 - \frac{1}{f_1} \right) = -f'_1 + \frac{f'_1}{(f_1)^2} = -\frac{f'_1(f_1-1)(f_1+1)}{(f_1)^2}.$$

We have $f'_1 < 0$ by the definition and we consider only the case $f_1(i) < 0$. Thus, the function $-(f_1+1)^2/f_1$ ($k(v-k)/v < i \leq k$) attains a minimum at i if $f_1(i) = -1$ that is

$$i = \alpha_1 = \frac{k(v-k)}{v-1}.$$

Finally we have the bound

$$\begin{aligned} |\mathcal{F}| &\geq \binom{v}{1} + \frac{v-1}{\binom{v}{1}} \min \left\{ \frac{(f_1(i)+1)^2}{-f_1(i)} \mid \frac{k(v-k)}{v} < i \leq k \ (i \in \mathbb{Z}) \right\} \\ &= \binom{v}{1} + \frac{v-1}{\binom{v}{1}} \min \left\{ \frac{(f_1(i)+1)^2}{-f_1(i)} \mid i = \lfloor \alpha_1 \rfloor, \lceil \alpha_1 \rceil, \frac{k(v-k)}{v} < i \leq k \right\} \end{aligned}$$

$$= \binom{v}{1} + \frac{v-1}{\binom{v}{2}} \min \left\{ \frac{(f_1(i) + 1)^2}{-f_1(i)} \mid i = \lfloor \alpha_1 \rfloor, \lceil \alpha_1 \rceil, \frac{k(v-k)}{v} < i \right\} \text{ since } \alpha_1 \leq k.$$

In the same way, if $v < k^2$ and $e = 2$, then the bound (\star) is

$$\begin{aligned} |\mathcal{F}| &\geq \binom{v}{2} + \frac{v-1}{\binom{v}{2}} \min \left\{ \frac{\left(\sum_{l=0}^2 Q_{i,l} \right)^2}{-Q_{i,1}} \mid \frac{k(v-k)}{v} < i \leq k \ (i \in \mathbb{Z}) \right\} \\ &= \binom{v}{2} + \frac{v-1}{\binom{v}{2}} \min \left\{ \frac{(f_2(i) + f_1(i) + 1)^2}{-f_1(i)} \mid \frac{k(v-k)}{v} < i \leq k \ (i \in \mathbb{Z}) \right\}. \end{aligned}$$

For $-(f_2 + f_1 + 1)^2/f_1$, we have

$$\begin{aligned} \frac{d}{di} \left(\frac{(f_2 + f_1 + 1)^2}{-f_1} \right) &= \frac{f_2 + f_1 + 1}{(f_1)^2} \{-2(f'_2 + f'_1)f_1 + (f_2 + f_1 + 1)f'_1\} \\ &= \frac{f_2 + f_1 + 1}{(f_1)^2} (-2f_1f'_2 - f_1f'_1 + f'_1f_2 + f'_1). \end{aligned}$$

We have $-2f_1f'_2 - f_1f'_1 + f'_1f_2 + f'_1 < 0$ for all i by the direct calculation. Thus, the function $-(f_2 + f_1 + 1)^2/f_1$ ($k(v-k)/v < i \leq k$) attains a minimum at i if $f_2(i) + f_1(i) + 1 = 0$ that is $i = \alpha_2$ or β_2 . By $(***)$, we have only $i = \alpha_2$ for a minimum value.

Finally we have the bound

$$\begin{aligned} |\mathcal{F}| &\geq \binom{v}{2} + \frac{v-1}{\binom{v}{2}} \min \left\{ \frac{(f_2(i) + f_1(i) + 1)^2}{-f_1(i)} \mid \frac{k(v-k)}{v} < i \leq k \ (i \in \mathbb{Z}) \right\} \\ &= \binom{v}{2} + \frac{v-1}{\binom{v}{2}} \min \left\{ \frac{(f_2(i) + f_1(i) + 1)^2}{-f_1(i)} \mid i = \lfloor \alpha_2 \rfloor, \lceil \alpha_2 \rceil, \frac{k(v-k)}{v} < i \leq k \right\} \\ &= \binom{v}{2} + \frac{v-1}{\binom{v}{2}} \min \left\{ \frac{(f_2(i) + f_1(i) + 1)^2}{-f_1(i)} \mid i = \lfloor \alpha_2 \rfloor, \lceil \alpha_2 \rceil, \frac{k(v-k)}{v} < i \right\}. \end{aligned}$$

Here we use $\alpha_2 \leq k$ from $(***)$.

(iii) If $e = 1$, then the bound $(\star\star)$ is

$$\begin{aligned} |\mathcal{F}| &\geq \binom{v}{1} + \frac{v(v-3)}{2\binom{v}{1}} \min \left\{ \frac{\left(\sum_{l=0}^1 Q_{i,l} \right)^2}{-Q_{i,2}} \mid \beta' < i < \alpha' \ (i \in \mathbb{Z}) \right\} \\ &= v + \frac{v-3}{2} \min \left\{ \frac{(f_1(i) + 1)^2}{-f_2(i)} \mid \beta' < i < \alpha' \ (i \in \mathbb{Z}) \right\}. \end{aligned}$$

For $-(f_1 + 1)^2/f_2$, we have

$$\begin{aligned} \frac{d}{di} \left(\frac{(f_1 + 1)^2}{-f_2} \right) &= \frac{f_1 + 1}{(f_2)^2} \{-2f'_1f_2 + (f_1 + 1)f'_2\} \\ &= \frac{f_1 + 1}{(f_1)^2} (-2f'_1f_2 + f_1f'_2 + f'_2). \end{aligned}$$

For any $i \leq k$, we have $-2f'_1f_2 + f_1f'_2 + f'_2 < 0$ by the direct calculation. Thus, the function $-(f_1 + 1)^2/f_2$ ($\beta' < i \leq \alpha'$) attains a minimum at i if $f_1(i) = -1$ that is

$$i = \alpha_1 = \frac{k(v-k)}{v-1}.$$

Finally we have the bound

$$|\mathcal{F}| \geq v + \frac{v-3}{2} \min \left\{ \frac{(f_1(i) + 1)^2}{-f_2(i)} \mid \beta' < i < \alpha' \ (i \in \mathbb{Z}) \right\}$$

$$\begin{aligned}
 &= v + \frac{v-3}{2} \min \left\{ \frac{(f_1(i)+1)^2}{-f_2(i)} \mid i = \lfloor \alpha_1 \rfloor, \lceil \alpha_1 \rceil, \beta' < i < \alpha' \right\}. \\
 &= v + \frac{v-3}{2} \min \left\{ \frac{(f_1(i)+1)^2}{-f_2(i)} \mid i = \lfloor \alpha_1 \rfloor, \lceil \alpha_1 \rceil, i < \alpha' \right\} \quad \text{since } \beta' < \lfloor \alpha_1 \rfloor.
 \end{aligned}$$

□

Chapter 4

Finite Projective Geometry

In this chapter, we focus on a finite projective geometry and determine the structure of the coherent configuration based on it. Dunkl [8] have worked in this field and calculated essential tools which are much helpful for our calculations. As a result of this work, we will present two applications which are for the Terwilliger algebras of Grassmann graphs and for the Erdős-Ko-Rado theorem for singular linear spaces.

4.1 Notations

We start this chapter with some notations. Let q be a positive integer with $q \neq 1$. For a nonnegative integer n , define the q -Pochhammer symbol:

$$(a; q)_n = \begin{cases} \prod_{j=0}^{n-1} (1 - aq^j) & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases} \quad (4.1)$$

For convenience, we will write $(a)_n = (a; q^{-1})_n$. Define the q -polynomial coefficient for nonnegative integers n and k by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{(q^n)_k}{(q^k)_k} & \text{if } k \leq n, \\ 0 & \text{otherwise.} \end{cases} \quad (4.2)$$

Remark that if q is a prime power, the value $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the number of k -dimensional subspaces of an n -dimensional vector space over the finite field \mathbb{F}_q . For parameters a, b, c , define the q -Hahn polynomial of degree m in q^x by

$$E_m(a, b, c, x; q^{-1}) = q^{m(m-1)/2} (q^a)_m (q^c)_m \sum_{j=0}^m \frac{(q^m)_j (q^{a+b-m+1})_j (q^x)_j}{(q^a)_j (q^c)_j (q^{-1})_j} q^{-j}. \quad (4.3)$$

For parameters a, b , define the q -Krawtchouk polynomial of degree m by

$$K_m(x; a, b; q) = \frac{(q^a)_m (q^b)_m}{(q^{-1})_m} q^{-m} \sum_{j=0}^m \frac{(q^{-m}; q)_j (q^{-x}; q)_j q^j}{(q^{-a}; q)_j (q^{-b}; q)_j (q; q)_j}. \quad (4.4)$$

4.2 The Coherent Configuration Based on a Finite Projective Geometry

A finite projective geometry X is the set of all subspaces of a finite dimensional vector space V over a finite field \mathbb{F}_q . Let V_a be a fixed subspace of V with dimension a and the dimension of V is denoted by $a + b$. We construct a coherent configuration on X from a group action. Let G be the following subgroup of the general linear group $\mathrm{GL}(a + b, q)$, the set of all invertible matrices of size $a + b$ over \mathbb{F}_q :

$$G = \{g \in \mathrm{GL}(a + b, q) \mid gV_a = V_a\}. \quad (4.5)$$

Let \mathcal{R} be the totality of G -orbits, where we consider the natural G -action on the Cartesian product $X \times X$. By Theorem 1.1, the pair (X, \mathcal{R}) forms a coherent configuration and we denote it by \mathfrak{X} . In this section, we consider the structure of the coherent configuration \mathfrak{X} .

Resulting from Dunkl [8], we introduce a different description of X . Let $V = V_a \oplus V_b$, where V_b has dimension b . For any two vector spaces W_1 and W_2 , $B(W_1, W_2)$ denotes the set of all linear maps from W_1 to W_2 .

Theorem 4.1. *There exists a bijective mapping between X and the set of triples (ξ_1, ξ_2, T) where ξ_1, ξ_2 are subspaces of V_a and V_b respectively, and $T \in B(\xi_2, V_a/\xi_1)$.*

Proof. For given $\xi \in X$, define

$$\xi_1 = \left\{ \mathbf{x}_1 \in V_a \mid \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{0} \end{pmatrix} \in \xi \right\}, \quad \xi_2 = \left\{ \mathbf{x}_2 \in V_b \mid \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} \in \xi \text{ for some } \mathbf{x}_1 \in V_a \right\}.$$

For $\mathbf{x}_2 \in \xi_2$, define a map $T \in B(\xi_2, V_a/\xi_1)$ by

$$T(\mathbf{x}_2) = \left\{ \mathbf{x}_1 \in V_a \mid \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} \in \xi \right\}.$$

This map is well-defined because $\mathbf{x}_1 + \mathbf{x}'_1 \in T(\mathbf{x}_2)$ for any $\mathbf{x}_1 \in \xi_1, \mathbf{x}_2 \in \xi_2, \mathbf{x}'_1 \in T(\mathbf{x}_2)$.

On the other hand, for given such triple (ξ_1, ξ_2, T) , we can define

$$\xi = \left\{ \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} \mid \mathbf{x}_2 \in \xi_2, \mathbf{x}_1 \in T(\mathbf{x}_2) \right\} \in X.$$

□

For the rest of this paper, we identify X with the set of triples:

$$X = \{(\xi_1, \xi_2, T) \mid \xi_1 \subset V_a, \xi_2 \subset V_b, T \in B(\xi_2, V_a/\xi_1)\}. \quad (4.6)$$

Note that the dimension of $(\xi_1, \xi_2, T) \in X$ is the sum of the two dimensions of ξ_1, ξ_2 . Next, we consider the G -action on the set of triples.

Theorem 4.2. *We can identify the group G with the set of triples*

$$\left\{ (g_1, g_2, S) \in \mathrm{GL}(a, q) \times \mathrm{GL}(b, q) \times B(V_b, V_a) \mid \begin{pmatrix} g_1 & S \\ 0 & g_2 \end{pmatrix} \in \mathrm{GL}(a+b, q) \right\}.$$

For $(g_1, g_2, S) \in G$ and $(\xi_1, \xi_2, T) \in X$, we have

$$(g_1, g_2, S)(\xi_1, \xi_2, T) = (g_1\xi_1, g_2\xi_2, (g_1T + \pi(g_1\xi_1)S|_{\xi_2})g_2^{-1}),$$

where $\pi(g_1\xi_1)$ is the projective map from V_a to $V_a/(g_1\xi_1)$.

Proof. Let $(\xi'_1, \xi'_2, T') = (g_1, g_2, S)(\xi_1, \xi_2, T)$. Then we have

$$\begin{aligned} \xi'_1 &= \left\{ \mathbf{x}'_1 \in V_a \mid \begin{pmatrix} \mathbf{x}'_1 \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} g_1\mathbf{x}_1 + S\mathbf{x}_2 \\ g_2\mathbf{x}_2 \end{pmatrix}, \text{ for some } \mathbf{x}_2 \in \xi_2, \mathbf{x}_1 \in T(\mathbf{x}_2) \right\} \\ &= \{ \mathbf{x}'_1 \in V_a \mid \mathbf{x}'_1 = g_1\mathbf{x}_1, \text{ for some } \mathbf{x}_1 \in T(\mathbf{0}) = \xi_1 \} \\ &= g_1\xi_1, \\ \xi'_2 &= \left\{ \mathbf{x}'_2 \in V_b \mid \begin{pmatrix} \mathbf{x}'_1 \\ \mathbf{x}'_2 \end{pmatrix} = \begin{pmatrix} g_1\mathbf{x}_1 + S\mathbf{x}_2 \\ g_2\mathbf{x}_2 \end{pmatrix}, \text{ for some } \mathbf{x}_2 \in \xi_2, \mathbf{x}_1 \in T(\mathbf{x}_2) \right\} \\ &= \{ g_2\mathbf{x}_2 \mid \mathbf{x}_2 \in \xi_2, \mathbf{x}_1 \in T(\mathbf{x}_2) \} \\ &= g_2\xi_2, \end{aligned}$$

and for $\mathbf{x}'_2 \in \xi'_2$,

$$T(\mathbf{x}'_2) = \left\{ \mathbf{x}'_1 \in V_a \mid \begin{pmatrix} \mathbf{x}'_1 \\ \mathbf{x}'_2 \end{pmatrix} = \begin{pmatrix} g_1\mathbf{x}_1 + S\mathbf{x}_2 \\ g_2\mathbf{x}_2 \end{pmatrix}, \text{ for some } \mathbf{x}_2 \in \xi_2, \mathbf{x}_1 \in T(\mathbf{x}_2) \right\}$$

$$\begin{aligned}
&= \{g_1 \mathbf{x}_1 + S \mathbf{x}_2 \mid \mathbf{x}_2 \in \xi_2, \mathbf{x}_1 \in T(\mathbf{x}_2), \mathbf{x}'_2 = g_2 \mathbf{x}_2\} \\
&= \{g_1 \mathbf{x}_1 + S g_2^{-1} \mathbf{x}'_2 \mid \mathbf{x}_1 \in T(g_2^{-1} \mathbf{x}'_2)\} \\
&= g_1 T(g_2^{-1} \mathbf{x}'_2) + S g_2^{-1} \mathbf{x}'_2 \\
&= [(g_1 T + \pi(g_1 \xi_1) S|_{\xi_2}) g_2^{-1}](\mathbf{x}'_2).
\end{aligned}$$

□

Define the decomposition $X = \sqcup_{i,j} X_{i,j}$, where

$$X_{i,j} = \{(\xi_1, \xi_2, T) \in X \mid \dim \xi_1 = i, \dim \xi_2 = j\}, \quad (0 \leq i \leq a, 0 \leq j \leq b). \quad (4.7)$$

Each $X_{i,j}$ is called a singular linear space, which is also denoted by $\mathcal{M}_q(i+j, i; a+b, b)$.

For each i, j , we fix an element $(\zeta_i, \omega_j, 0) \in X_{i,j}$ satisfying

$$\{\mathbf{0}\} = \zeta_0 \subsetneq \zeta_1 \subsetneq \cdots \subsetneq \zeta_a = V_a, \quad \{\mathbf{0}\} = \omega_0 \subsetneq \omega_1 \subsetneq \cdots \subsetneq \omega_b = V_b. \quad (4.8)$$

Then we denote the stabilizer subgroup of $(\zeta_i, \omega_j, 0)$ in G by $H_{i,j}$. Note that we can identify $X_{i,j}$ with the cosets $G/H_{i,j}$. Here an element $(\xi_1, \xi_2, T) \in X_{i,j}$ corresponds to $gH_{i,j} \in G/H_{i,j}$ if $(\xi_1, \xi_2, T) = g(\zeta_i, \omega_j, 0)$.

4.2.1 Parts of the Coherent Configuration $\mathfrak{X} = (X, \mathcal{R})$

We divide the coherent configuration $\mathfrak{X} = (X, \mathcal{R})$ into the parts $\mathfrak{X}_{x,u,i,j} = (X_{x,u}, X_{i,j}, \mathcal{R}_{x,u,i,j})$ from the decomposition of X . Here each element of $\mathcal{R}_{x,u,i,j}$ is a subset of $X_{x,u} \times X_{i,j}$. From now on, we consider the parts $\mathfrak{X}_{x,u,i,j}$ instead of the whole \mathfrak{X} . Note that the diagonal parts $\mathfrak{X}_{i,j,i,j} = (X_{i,j}, X_{i,j}, \mathcal{R}_{i,j,i,j})$ are association schemes based on singular linear spaces $X_{i,j}$. Since $X_{i,j} = G/H_{i,j}$, the part $\mathfrak{X}_{x,u,i,j}$ is the structure obtained from the orbitals of the G -action on $(G/H_{x,u}) \times (G/H_{i,j})$, which we have worked in Chapter 2. In this subsection, we fix x, u, i, j such that $0 \leq x, i \leq a$ and $0 \leq u, j \leq b$ and focus only on one part $\mathfrak{X}_{x,u,i,j}$. For the following discussions, define the index set K by

$$K = \left\{ (v, w, z) \in \mathbb{Z}^3 \mid \begin{array}{l} \max\{0, -a + i + x\} \leq v \leq \min\{i, x\}, \\ \max\{0, -b + j + u\} \leq w \leq \min\{j, u\}, \\ 0 \leq z \leq \min\{w, a - i + v - x\} \end{array} \right\}. \quad (4.9)$$

Theorem 4.3. For $(v, w, z) \in K$, define

$$R_{v,w,z} = \left\{ (\xi_1, \xi_2, T), (\eta_1, \eta_2, S) \in X_{x,u} \times X_{i,j} \mid \begin{array}{l} \dim(\xi_1 \cap \eta_1) = v, \\ \dim(\xi_2 \cap \eta_2) = w, \\ \text{rank}(\pi(\xi_1 + \eta_1) T|_{\xi_2 \cap \eta_2} - \pi(\xi_1 + \eta_1) S|_{\xi_2 \cap \eta_2}) = z \end{array} \right\}. \quad (4.10)$$

Then $\{R_{v,w,z}\}_{(v,w,z) \in K}$ covers the part $\mathcal{R}_{x,u,i,j}$ of the totality of G -orbits \mathcal{R} . Here $\pi(\xi_1 + \eta_1)$ is the projective map from V_a to $V_a/(\xi_1 + \eta_1)$.

In particular, if we see $R_{v,w,z}$ as a subset of $(G/H_{x,u}) \times (G/H_{i,j})$, we have

$$(H_{x,u}, H_{i,j}) \in R_{\min\{x,i\}, \min\{u,j\}, 0}.$$

Proof. We divide the proof into three parts: (i) the $R_{v,w,z}$ give a partition of $X_{x,u} \times X_{i,j}$, (ii) each of the $R_{v,w,z}$ is closed under the G -action, (iii) for any two elements in $R_{v,w,z}$, there exists an element in G which maps one of them to the other.

- (i) If $(v, w, z) \neq (v', w', z') \in K$, it is clear, from the definition, that $R_{v,w,z} \cap R_{v',w',z'} = \emptyset$. It is also clear that, we have

$$0 \leq \dim(\xi_1 \cap \eta_1) \leq \min\{x, i\}.$$

In addition we have the dimension formula:

$$\dim(\xi_1 \cap \eta_1) = \dim \xi_1 + \dim \eta_1 - \dim(\xi_1 + \eta_1) \geq x + i - a.$$

In the same way, we obtain

$$\max\{0, u + j - b\} \leq \dim(\xi_2 \cap \eta_2) \leq \min\{u, j\}.$$

Lastly, we have

$$\begin{aligned} \text{rank}(\pi(\xi_1 + \eta_1)T|_{\xi_2 \cap \eta_2} - \pi(\xi_1 + \eta_1)S|_{\xi_2 \cap \eta_2}) &\leq \min\{V_a - \dim(\xi_1 + \eta_1), \dim(\xi_2 \cap \eta_2)\} \\ &= \min\{a - (x + i - v), w\}. \end{aligned}$$

(ii) The G -action preserves $\dim(\xi_1 \cap \eta_1)$ and $\dim(\xi_2 \cap \eta_2)$. For (g_1, g_2, S_0) , we have

$$\begin{aligned} &\text{rank}(\pi(\xi_1 + \eta_1)T|_{\xi_2 \cap \eta_2} - \pi(\xi_1 + \eta_1)S|_{\xi_2 \cap \eta_2}) \\ &= \text{rank}(\pi(g_1\xi_1 + g_1\eta_1)g_1Tg_2^{-1}|_{g_2\xi_2 \cap g_2\eta_2} - \pi(g_1\xi_1 + g_1\eta_1)g_1Sg_2^{-1}|_{g_2\xi_2 \cap g_2\eta_2}) \\ &= \text{rank}(\pi(g_1\xi_1 + g_1\eta_1)(g_1T + \pi(g_1\xi_1)S_0|_{\xi_2})g_2^{-1}|_{g_2\xi_2 \cap g_2\eta_2} \\ &\quad - \pi(g_1\xi_1 + g_1\eta_1)(g_1S + \pi(g_1\eta_1)S_0|_{\eta_2})g_2^{-1}|_{g_2\xi_2 \cap g_2\eta_2}). \end{aligned}$$

This tells us that $\text{rank}(\pi(\xi_1 + \eta_1)T|_{\xi_2 \cap \eta_2} - \pi(\xi_1 + \eta_1)S|_{\xi_2 \cap \eta_2})$ is also preserved.

(iii) Let $((\xi_1, \xi_2, T), (\eta_1, \eta_2, S))$ and $((\xi'_1, \xi'_2, T'), (\eta'_1, \eta'_2, S'))$ be two elements in $R_{v,w,z}$. We construct an element in G which maps the first one to the second one. First we can take g_1, g_2 such that $g_i\xi_i = \xi'_i$ and $g_i\eta_i = \eta'_i$ for $i = 1, 2$. Then we may assume that $\xi'_i = \xi_i$ and $\eta'_i = \eta_i$ for $i = 1, 2$. Next, by taking suitable g_1, g_2 , we may assume that

$$\pi(\xi_1 + \eta_1)T|_{\xi_2 \cap \eta_2} - \pi(\xi_1 + \eta_1)S|_{\xi_2 \cap \eta_2} = \pi(\xi_1 + \eta_1)T'|_{\xi_2 \cap \eta_2} - \pi(\xi_1 + \eta_1)S'|_{\xi_2 \cap \eta_2},$$

since both have the same rank. Thus, we can take S_0 such that

$$T' = T + \pi(\xi_1)S_0|_{\xi_2},$$

$$S' = S + \pi(\eta_1)S_0|_{\eta_2}.$$

Then $(e_1, e_2, S_0) \in G$ maps the first one to the second one, where $e_1 \in \text{GL}(a, q)$ and $e_2 \in \text{GL}(b, q)$ are the identity elements

□

The next theorem will be used later on.

Theorem 4.4. For any $(v, w, z) \in K$ and $((\xi_1, \xi_2, T), (\eta_1, \eta_2, S)) \in R_{v,w,z}$, we have

$$\dim((\xi_1, \xi_2, T) \cap (\eta_1, \eta_2, S)) = v + w - z.$$

Proof. We have

$$\begin{aligned} &\dim((\xi_1, \xi_2, T) \cap (\eta_1, \eta_2, S)) \\ &= \dim \left\{ \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} \mid \mathbf{x}_2 \in \xi_2, \mathbf{x}_1 \in T(\mathbf{x}_2) \right\} \cap \left\{ \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} \mid \mathbf{x}_2 \in \eta_2, \mathbf{x}_1 \in S(\mathbf{x}_2) \right\} \\ &= \dim \left\{ \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} \mid \mathbf{x}_2 \in \xi_2 \cap \eta_2, \mathbf{x}_1 \in T(\mathbf{x}_2) \cap S(\mathbf{x}_2) \right\} \\ &= \dim \left\{ \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} \mid \mathbf{x}_2 \in \text{Ker}(\pi(\xi_1 + \eta_1)T|_{\xi_2 \cap \eta_2} - \pi(\xi_1 + \eta_1)S|_{\xi_2 \cap \eta_2}), \mathbf{x}_1 \in T(\mathbf{x}_2) \cap S(\mathbf{x}_2) \right\} \\ &= (w - z) + v. \end{aligned}$$

□

Since $\mathfrak{X}_{x,u,i,j}$ is not a commutative association scheme in general, we define the structure “the adjacency matrices” and “the primitive idempotents” in terms of Section 2.2. Also we will introduce “the first and second eigenmatrices” in this terminology.

For $(v, w, z) \in K$, let “the adjacency matrix” $A_{v,w,z}$ be the $\{0, 1\}$ -matrix defined by $R_{v,w,z}$ indexed by $X_{x,u} \times X_{i,j}$:

$$(A_{v,w,z})_{\xi,\eta} = \begin{cases} 1 & \text{if } (\xi, \eta) \in R_{v,w,z}, \\ 0 & \text{otherwise,} \end{cases} \quad (\xi \in X_{x,u}, \eta \in X_{i,j}). \quad (4.11)$$

We also see this as a linear map from $L(X_{i,j})$ to $L(X_{x,u})$.

For “the primitive idempotents”, we need to consider two decompositions of $L(X_{i,j})$ and $L(X_{x,u})$, which are of vital importance and mainly worked due to Dunkl [8]. Before stating Dunkl’s result, we prepare some notations. Define the set of triples:

$$Y = \left\{ (\eta_1, \eta_2, S) \mid \begin{array}{l} \eta_1 \subset V_a, \eta_2 \subset V_b, S \in B(V_a, V_b), \\ \eta_1 \subset \text{Ker } S, \text{Im } S \subset \eta_2 \end{array} \right\} \quad (4.12)$$

Here we see $(\eta_1, \eta_2, S) \in Y$ as a function on X by setting the value at $(\xi_1, \xi_2, T) \in X$ by

$$(\eta_1, \eta_2, S)((\xi_1, \xi_2, T)) = \begin{cases} \chi(\text{Tr}(ST)) & \text{if } \eta_1 \subset \xi_1 \subset \text{Ker } S \text{ and } \eta_2 \subset \xi_2, \\ 0 & \text{otherwise.} \end{cases} \quad (4.13)$$

Dunkl [8] showed that Y is a basis of $L(X)$. Define linear operators D_1, D_2 on $L(X)$ by

$$D_1((\eta_1, \eta_2, S)) = \sum_{\substack{\omega_1 \subset \eta_1 \\ \dim \omega_1 = \dim \eta_1 - 1}} (\omega_1, \eta_2, S) \quad ((\eta_1, \eta_2, S) \in Y), \quad (4.14)$$

$$D_2((\eta_1, \eta_2, S)) = \sum_{\substack{\text{Im } S \subset \omega_2 \subset \eta_2 \\ \dim \omega_2 = \dim \eta_2 - 1}} (\eta_1, \omega_2, S) \quad ((\eta_1, \eta_2, S) \in Y). \quad (4.15)$$

For nonnegative integers m, n, r such that $2m + r \leq a, 2n + r \leq b$, define

$$P_{m,n,r} = \langle (\eta_1, \eta_2, S) \in Y \mid \dim \eta_1 = m, \dim \eta_2 = n + r, \text{rank } S = r \rangle. \quad (4.16)$$

Here the angle bracket means the linear span in $L(X)$. For each m, n, r , define

$$V_{m,n,r} = P_{m,n,r} \cap \text{Ker } D_1 \cap \text{Ker } D_2. \quad (4.17)$$

Dunkl [8] showed that the $V_{m,n,r}$ are irreducible G -submodules of $L(X)$. Again, we define the index sets for the following discussions:

$$L_{(x,u)} = \left\{ (m, n, r) \in \mathbb{Z}^3 \mid \begin{array}{l} 0 \leq r \leq \min\{a - x, u\} \\ 0 \leq m \leq \min\{x, a - r - x\} \\ 0 \leq n \leq \min\{u - r, b - u\} \end{array} \right\}. \quad (4.18)$$

We also define $L_{(i,j)}$ by replacing (x, u) by (i, j) and define

$$L = L_{(x,u)} \cap L_{(i,j)}. \quad (4.19)$$

We have a remark about K and L .

Proposition 4.5. *The two index sets K, L have the same cardinality.*

Proof. Set $s = \min\{i, x\}$ and $t = \min\{j, u\}$. We show that the mapping $(v, w, z) \mapsto (m, n, r) = (s - v, t - w, z)$ is bijective.

$$K = \left\{ (v, w, z) \in \mathbb{Z}^3 \mid \begin{array}{l} \max\{0, -a + i + x\} \leq v \leq \min\{i, x\}, \\ \max\{0, -b + j + u\} \leq w \leq \min\{j, u\}, \\ 0 \leq z \leq \min\{w, a - i + v - x\} \end{array} \right\}$$

$$\begin{aligned}
&= \left\{ (v, w, z) \in \mathbb{Z}^3 \mid \begin{array}{l} 0 \leq s - v \leq \min\{i, x, a - x, a - i\}, \\ 0 \leq t - w \leq \min\{j, u, b - u, b - j\}, \\ 0 \leq z \leq \min\{t - (t - w), a - i - x + s - (s - v)\} \end{array} \right\} \\
&= \left\{ (v, w, z) \in \mathbb{Z}^3 \mid \begin{array}{l} 0 \leq s - v \leq \min\{i, x, a - x, a - i\}, \\ 0 \leq t - w \leq \min\{j, u, b - u, b - j\}, \\ 0 \leq z \leq \min\{t, a - i - x + s\}, \\ z \leq t - (t - w), \\ z \leq a - i - x + s - (s - v) \end{array} \right\} \\
&= \left\{ (v, w, z) \in \mathbb{Z}^3 \mid \begin{array}{l} 0 \leq s - v \leq \min\{i, x, a - x, a - i\}, \\ 0 \leq t - w \leq \min\{j, u, b - u, b - j\}, \\ 0 \leq z \leq \min\{t, a - i - x + s\}, \\ t - w \leq t - z, \\ s - v \leq a - i - x + s - z \end{array} \right\} \\
&= \left\{ (v, w, z) \in \mathbb{Z}^3 \mid \begin{array}{l} 0 \leq s - v \leq \min\{i, x, a - x, a - i\}, \\ 0 \leq t - w \leq \min\{j, u, b - u, b - j\}, \\ 0 \leq z \leq \min\{u, j, a - i, a - x\}, \\ t - w \leq \min\{u - z, j - z\}, \\ s - v \leq \min\{a - i - z, a - x - z\} \end{array} \right\} \\
&= \left\{ (v, w, z) \in \mathbb{Z}^3 \mid \begin{array}{l} 0 \leq s - v \leq \min\{i, x, a - x - z, a - i - z\}, \\ 0 \leq t - w \leq \min\{j - z, u - z, b - u, b - j\}, \\ 0 \leq z \leq \min\{u, j, a - i, a - x\} \end{array} \right\} \\
&\rightarrow L.
\end{aligned}$$

□

Theorem 4.6 (Dunkl [8]). *We have the following:*

(i) *For nonnegative integers m, n, r such that $2m + r \leq a$ and $2n + r \leq b$, we have*

$$\dim V_{m,n,r} = \frac{(q^a)_r (q^b)_r}{(q^{-1})_r} q^{-r} \begin{bmatrix} a - r \\ m \end{bmatrix}_q \begin{bmatrix} b - r \\ n \end{bmatrix}_q \left(\frac{q^m - q^{a-r-m+1}}{1 - q^{a-r-m+1}} \right) \left(\frac{q^n - q^{b-r-n+1}}{1 - q^{b-r-n+1}} \right). \quad (4.20)$$

(ii) *We define the G -module $V_{m,n,r}^{i,j} = \{f|_{X_{i,j}} \mid f \in V_{m,n,r}\}$ and also define $V_{m,n,r}^{x,u}$ in the same manner. Then for $(m, n, r) \in L$, we have $V_{m,n,r}^{i,j} \simeq V_{m,n,r}^{x,u}$ ($\simeq V_{m,n,r}$).*

(iii) *We have the irreducible decomposition $L(X_{x,u}) = \bigoplus_{(m,n,r) \in L(x,u)} V_{m,n,r}^{x,u}$. We also have the irreducible decomposition for $L(X_{i,j})$ in the same manner.*

(iv) *For $(m, n, r) \in L$, the space $(V_{m,n,r}^{x,u})^{H_{i,j}} = \{f \in V_{m,n,r}^{x,u} \mid H_{i,j}\text{-invariant}\}$ is spanned by one element whose value at $(\xi_1, \xi_2, T) \in X_{x,u}$ is*

$$\frac{E_m(i, a - r - i, x, v; q^{-1}) E_n(j - r, b - j, u - r, w - r; q^{-1}) K_r(z; w, a - x - i + v; q)}{E_m(i, a - r - i, x, s; q^{-1}) E_n(j - r, b - j, u - r, t - r; q^{-1}) K_r(0; t, a - x - i + s; q)}, \quad (4.21)$$

where $s = \min\{i, x\}$ and $t = \min\{j, u\}$, $((\xi_1, \xi_2, T), (\zeta_i, \omega_j, 0)) \in R_{v,w,z}$. In particular, the function (4.21) is the spherical function in $V_{m,n,r}^{x,u}$.

For all $(m, n, r) \in L$, fix isomorphism maps $\pi_{m,n,r} : V_{m,n,r}^{i,j} \rightarrow V_{m,n,r}^{x,u}$ where $\pi_{m,n,r}(f|_{X_{i,j}}) = f|_{X_{x,u}}$ ($f \in V_{m,n,r}$). We define “the primitive idempotent” $E_{m,n,r} : L(X_{i,j}) \rightarrow L(X_{x,u})$ for any $(m, n, r) \in L$:

$$E_{m,n,r}|_{V_{m',n',r'}^{i,j}} = \begin{cases} \pi_{m,n,r} & \text{if } (m, n, r) = (m', n', r'), \\ 0 & \text{otherwise.} \end{cases} \quad (4.22)$$

Similarly, the map $(E_{m,n,r})' : L(X_{x,u}) \rightarrow L(X_{i,j})$ should be defined as

$$(E_{m,n,r})'|_{V_{m',n',r'}^{x,u}} := \begin{cases} \pi_{m,n,r}^{-1} & \text{if } (m, n, r) = (m', n', r'), \\ 0 & \text{otherwise.} \end{cases} \quad (4.23)$$

Finally, we define “the first and second eigenmatrices” of $\mathfrak{X}_{x,u,i,j}$:

$$(A_{v,w,z})^T = \sum_{(m,n,r) \in L} P_{(m,n,r),(v,w,z)}^{x,u,i,j} (E_{m,n,r})', \quad E_{m,n,r} = \frac{1}{|R_{s,t,0}|} \sum_{(v,w,z) \in K} Q_{(v,w,z),(m,n,r)}^{x,u,i,j} A_{v,w,z}. \quad (4.24)$$

For a remark, the matrices $\mathcal{P}^{x,u,i,j} = (P_{(m,n,r),(v,w,z)}^{x,u,i,j})$, $\mathcal{Q}^{x,u,i,j} = (Q_{(v,w,z),(m,n,r)}^{x,u,i,j})$ are the first and second eigenmatrices for the association scheme $\mathfrak{X}_{i,j,i,j}$ based on singular linear space if $x = i$ and $u = j$.

4.2.2 Matrices $\mathcal{P}^{x,u,i,j}$ and $\mathcal{Q}^{x,u,i,j}$

In this subsection, we also fix x, u, i, j with $0 \leq x, i \leq a$ and $0 \leq u, j \leq b$ and set integers $s = \min\{i, x\}$ and $t = \min\{j, u\}$. To determine the matrices $\mathcal{P}^{x,u,i,j}$ and $\mathcal{Q}^{x,u,i,j}$, we use the general formula in Chapter 2. In particular, we use Theorem 2.5 and Theorem 2.7 which says

$$P_{(m,n,r),(v,w,z)}^{x,u,i,j} = \frac{|R_{v,w,z}|}{|R_{s,t,0}|} P_{(m,n,r),(s,t,0)}^{x,u,i,j} \omega_{m,n,r}(v, w, z), \quad (4.25)$$

$$Q_{(v,w,z),(m,n,r)}^{x,u,i,j} = \frac{|R_{s,t,0}|}{|R_{v,w,z}|} (\dim V_{m,n,r}) P_{(m,n,r),(v,w,z)}^{x,u,i,j}. \quad (4.26)$$

Here $\omega_{m,n,r}(v, w, z)$ is namely the value of the spherical function given in (4.21). Thus, our goal of this subsection is to determine the values of the right-hand side of each of (4.25) and (4.26).

First we calculate the special value $P_{(m,n,r),(s,t,0)}^{x,u,i,j}$ for any $(m, n, r) \in L$. Define operators L_1, L_2, R_1, R_2 on $L(X)$ by

$$L_1 f((\xi_1, \xi_2, T)) = \sum_{\substack{\omega_1 \supseteq \xi_1 \\ \dim \omega_1 = \dim \xi_1 + 1}} f((\omega_1, \xi_2, \pi(\omega_1)T)), \quad (4.27)$$

$$L_2 f((\xi_1, \xi_2, T)) = \sum_{\substack{\omega_2 \supseteq \xi_2 \\ \dim \omega_2 = \dim \xi_2 + 1}} \sum_{\substack{T_1 \in B(\omega_2, V_a / \xi_1) \\ T_1|_{\xi_2} = T}} f((\xi_1, \omega_2, T_1)), \quad (4.28)$$

$$R_1 f((\xi_1, \xi_2, T)) = \sum_{\substack{\omega_1 \subseteq \xi_1 \\ \dim \omega_1 = \dim \xi_1 - 1}} \sum_{\substack{T_1 \in B(\xi_2, V_a / \omega_1) \\ \pi(\xi_1)T_1 = T}} f((\omega_1, \xi_2, T_1)), \quad (4.29)$$

$$R_2 f((\xi_1, \xi_2, T)) = \sum_{\substack{\omega_2 \subseteq \xi_2 \\ \dim \omega_2 = \dim \xi_2 - 1}} f((\xi_1, \omega_2, T|_{\omega_2})). \quad (4.30)$$

“The adjacency matrix” $A_{s,t,0}$ can be expressed in terms of these operators.

Lemma 4.7. *We have*

$$A_{s,t,0} = \begin{cases} \left(\prod_{n=1}^{i-x} [n]_q \right)^{-1} (L_1)^{i-x} |_{L(X_{i,j})} & \text{if } x < i, u = j, \\ \left(\prod_{n=1}^{j-u} [n]_q \right)^{-1} (L_2)^{j-u} |_{L(X_{i,j})} & \text{if } x = i, u < j, \\ \left(\prod_{n=1}^{x-i} [n]_q \right)^{-1} (R_1)^{x-i} |_{L(X_{i,j})} & \text{if } x > i, u = j, \\ \left(\prod_{n=1}^{u-j} [n]_q \right)^{-1} (R_2)^{u-j} |_{L(X_{i,j})} & \text{if } x = i, u > j. \end{cases}$$

Proof. Let $f \in L(X_{i,j})$, $(\xi_1, \xi_2, T) \in X_{x,u}$.

(i) Let $x < i$ and $u = j$. We prove by induction on $(i - x)$. Suppose $i - x = 1$, then

$$[A_{s,t,0}f](\xi_1, \xi_2, T) = \sum_{\substack{\xi_1 \subset \eta_1 \subset V_a \\ \dim \eta_1 = x+1}} \sum_{\substack{S \in B(\xi_2, V_a / \eta_1) \\ \pi(\eta_1)T = S}} f(\eta_1, \xi_2, S) = [L_1 f](\xi_1, \xi_2, T).$$

Assume $i - x > 1$, we have

$$\begin{aligned} [A_{s,t,0}f](\xi_1, \xi_2, T) &= \sum_{\substack{\xi_1 \subset \eta_1 \subset V_a \\ \dim \eta_1 = i}} \sum_{\substack{S \in B(\xi_2, V_a / \eta_1) \\ \pi(\eta_1)T = S}} f(\eta_1, \xi_2, S) \\ &= \sum_{\substack{\xi_1 \subset \eta_1 \subset V_a \\ \dim \eta_1 = i}} f(\eta_1, \xi_2, \pi(\eta_1)T) \\ &= \begin{bmatrix} i-x \\ 1 \end{bmatrix}_q^{-1} \sum_{\substack{\xi_1 \subset \eta_1 \subset V_a \\ \dim \eta_1 = i}} \sum_{\substack{\xi_1 \subset \bar{\eta}_1 \subset \eta_1 \\ \dim \bar{\eta}_1 = i-1}} f(\eta_1, \xi_2, \pi(\eta_1)T) \\ &= \begin{bmatrix} i-x \\ 1 \end{bmatrix}_q^{-1} \sum_{\substack{\xi_1 \subset \eta_1 \subset V_a \\ \dim \eta_1 = i}} \sum_{\substack{\xi_1 \subset \bar{\eta}_1 \subset \eta_1 \\ \dim \bar{\eta}_1 = i-1}} f(\eta_1, \xi_2, \pi(\eta_1)\pi(\bar{\eta}_1)T) \\ &= \begin{bmatrix} i-x \\ 1 \end{bmatrix}_q^{-1} \sum_{\substack{\xi_1 \subset \bar{\eta}_1 \subset V_a \\ \dim \bar{\eta}_1 = i-1}} \sum_{\substack{\bar{\eta}_1 \subset \eta_1 \subset V_a \\ \dim \eta_1 = i}} f(\eta_1, \xi_2, \pi(\eta_1)\pi(\bar{\eta}_1)T) \\ &= \begin{bmatrix} i-x \\ 1 \end{bmatrix}_q^{-1} \sum_{\substack{\xi_1 \subset \bar{\eta}_1 \subset V_a \\ \dim \bar{\eta}_1 = i-1}} [L_1 f](\bar{\eta}_1, \xi_2, \pi(\bar{\eta}_1)T) \\ &= \begin{bmatrix} i-x \\ 1 \end{bmatrix}_q^{-1} \left(\prod_{n=1}^{i-x-1} \begin{bmatrix} n \\ 1 \end{bmatrix}_q \right)^{-1} [(L_1)^{i-x} f](\xi_1, \xi_2, T). \end{aligned}$$

(ii) Let $x = i$ and $u < j$. We prove by induction on $(j - u)$. Suppose $j - u = 1$, then

$$[A_{s,t,0}f](\xi_1, \xi_2, T) = \sum_{\substack{\xi_2 \subset \eta_2 \subset V_b \\ \dim \eta_2 = u+1}} \sum_{\substack{S \in B(\eta_2, V_a / \xi_1) \\ S|_{\xi_2} = T}} f(\xi_1, \eta_2, S) = [L_2 f](\xi_1, \xi_2, T).$$

Assume $j - u > 1$, we have

$$\begin{aligned} [A_{s,t,0}f](\xi_1, \xi_2, T) &= \sum_{\substack{\xi_2 \subset \eta_2 \subset V_b \\ \dim \eta_2 = j}} \sum_{\substack{S \in B(\eta_2, V_a / \xi_1) \\ S|_{\xi_2} = T}} f(\xi_1, \eta_2, S) \\ &= \begin{bmatrix} j-u \\ 1 \end{bmatrix}_q^{-1} \sum_{\substack{\xi_2 \subset \eta_2 \subset V_b \\ \dim \eta_2 = j}} \sum_{\substack{\xi_2 \subset \bar{\eta}_2 \subset \eta_2 \\ \dim \bar{\eta}_2 = j-1}} \sum_{\substack{S \in B(\eta_2, V_a / \xi_1) \\ S|_{\xi_2} = T}} f(\xi_1, \eta_2, S) \\ &= \begin{bmatrix} j-u \\ 1 \end{bmatrix}_q^{-1} \sum_{\substack{\xi_2 \subset \bar{\eta}_2 \subset V_b \\ \dim \bar{\eta}_2 = j-1}} \sum_{\substack{\bar{\eta}_2 \subset \eta_2 \subset V_b \\ \dim \eta_2 = j}} \sum_{\substack{S \in B(\eta_2, V_a / \xi_1) \\ S|_{\xi_2} = T}} f(\xi_1, \eta_2, S) \\ &= \begin{bmatrix} j-u \\ 1 \end{bmatrix}_q^{-1} \sum_{\substack{\xi_2 \subset \bar{\eta}_2 \subset V_b \\ \dim \bar{\eta}_2 = j-1}} \sum_{\substack{\bar{\eta}_2 \subset \eta_2 \subset V_b \\ \dim \eta_2 = j}} [L_2 f](\xi_1, \bar{\eta}_2, \bar{S}) \\ &= \begin{bmatrix} j-u \\ 1 \end{bmatrix}_q^{-1} \left(\prod_{n=1}^{j-u-1} \begin{bmatrix} n \\ 1 \end{bmatrix}_q \right)^{-1} [(L_2)^{j-u} f](\xi_1, \xi_2, T). \end{aligned}$$

(iii) Let $x > i$ and $u = j$. We prove by induction on $(x - i)$. Suppose $x - i = 1$, then

$$[A_{s,t,0}f](\xi_1, \xi_2, T) = \sum_{\substack{\eta_1 \subset \xi_1 \\ \dim \eta_1 = x-1}} \sum_{\substack{S \in B(\xi_2, V_a / \eta_1) \\ \pi(\xi_1)S = T}} f(\eta_1, \xi_2, S) = [R_1 f](\xi_1, \xi_2, T).$$

Assume $x - i > 1$, we have

$$\begin{aligned}
[A_{s,t,0}f](\xi_1, \xi_2, T) &= \sum_{\substack{\eta_1 \subset \xi_1 \\ \dim \eta_1 = i}} \sum_{\substack{S \in B(\xi_2, V_a / \eta_1) \\ \pi(\xi_1)S = T}} f(\eta_1, \xi_2, S) \\
&= \left[\begin{matrix} x-i \\ 1 \end{matrix} \right]_q^{-1} \sum_{\substack{\eta_1 \subset \xi_1 \\ \dim \eta_1 = i}} \sum_{\substack{\eta_1 \subset \bar{\eta}_1 \subset \xi_1 \\ \dim \bar{\eta}_1 = i+1}} \sum_{\substack{S \in B(\xi_2, V_a / \eta_1) \\ \pi(\xi_1)S = T}} f(\eta_1, \xi_2, S) \\
&= \left[\begin{matrix} x-i \\ 1 \end{matrix} \right]_q^{-1} \sum_{\substack{\bar{\eta}_1 \subset \xi_1 \\ \dim \bar{\eta}_1 = i+1}} \sum_{\substack{\eta_1 \subset \bar{\eta}_1 \\ \dim \eta_1 = i}} \sum_{\substack{S \in B(\xi_2, V_a / \bar{\eta}_1) \\ \pi(\xi_1)\bar{S} = T}} \sum_{\substack{S \in B(\xi_2, V_a / \eta_1) \\ \pi(\bar{\eta}_1)S = \bar{S}}} f(\eta_1, \xi_2, S) \\
&= \left[\begin{matrix} x-i \\ 1 \end{matrix} \right]_q^{-1} \sum_{\substack{\bar{\eta}_1 \subset \xi_1 \\ \dim \bar{\eta}_1 = i+1}} \sum_{\substack{\bar{S} \in B(\xi_2, V_a / \bar{\eta}_1) \\ \pi(\xi_1)\bar{S} = T}} [R_1 f](\bar{\eta}_1, \xi_2, \bar{S}) \\
&= \left[\begin{matrix} x-i \\ 1 \end{matrix} \right]_q^{-1} \left(\prod_{n=1}^{x-i-1} \left[\begin{matrix} n \\ 1 \end{matrix} \right]_q \right)^{-1} [(R_1)^{x-i} f](\xi_1, \xi_2, T).
\end{aligned}$$

(iv) Let $x = i$ and $u > j$. We prove by induction on $(u - j)$. Suppose $u - j = 1$, then

$$[A_{s,t,0}f](\xi_1, \xi_2, T) = \sum_{\substack{\eta_2 \subset \xi_2 \\ \dim \eta_2 = u-1}} \sum_{\substack{S \in B(\eta_2, V_a / \xi_1) \\ S = T|_{\eta_2}}} f(\xi_1, \eta_2, S) = [R_2 f](\xi_1, \xi_2, T).$$

Assume $u - j > 1$, we have

$$\begin{aligned}
[A_{s,t,0}f](\xi_1, \xi_2, T) &= \sum_{\substack{\eta_2 \subset \xi_2 \\ \dim \eta_2 = j}} \sum_{\substack{S \in B(\eta_2, V_a / \xi_1) \\ S = T|_{\eta_2}}} f(\xi_1, \eta_2, S) \\
&= \sum_{\substack{\eta_2 \subset \xi_2 \\ \dim \eta_2 = j}} f(\xi_1, \eta_2, T|_{\eta_2}) \\
&= \left[\begin{matrix} u-j \\ 1 \end{matrix} \right]_q^{-1} \sum_{\substack{\eta_2 \subset \xi_2 \\ \dim \eta_2 = j}} \sum_{\substack{\eta_2 \subset \bar{\eta}_2 \subset \xi_2 \\ \dim \bar{\eta}_2 = j+1}} f(\xi_1, \eta_2, T|_{\eta_2}) \\
&= \left[\begin{matrix} u-j \\ 1 \end{matrix} \right]_q^{-1} \sum_{\substack{\eta_2 \subset \xi_2 \\ \dim \eta_2 = j}} \sum_{\substack{\eta_2 \subset \bar{\eta}_2 \subset \xi_2 \\ \dim \bar{\eta}_2 = j+1}} f(\xi_1, \eta_2, (T|_{\bar{\eta}_2})|_{\eta_2}) \\
&= \left[\begin{matrix} u-j \\ 1 \end{matrix} \right]_q^{-1} \sum_{\substack{\bar{\eta}_2 \subset \xi_2 \\ \dim \bar{\eta}_2 = j+1}} \sum_{\substack{\eta_2 \subset \bar{\eta}_2 \\ \dim \eta_2 = j}} f(\xi_1, \eta_2, (T|_{\bar{\eta}_2})|_{\eta_2}) \\
&= \left[\begin{matrix} u-j \\ 1 \end{matrix} \right]_q^{-1} \sum_{\substack{\bar{\eta}_2 \subset \xi_2 \\ \dim \bar{\eta}_2 = j+1}} [R_2 f](\xi_1, \bar{\eta}_2, T|_{\bar{\eta}_2}) \\
&= \left[\begin{matrix} u-j \\ 1 \end{matrix} \right]_q^{-1} \left(\prod_{n=1}^{u-j-1} \left[\begin{matrix} n \\ 1 \end{matrix} \right]_q \right)^{-1} [(R_2)^{u-j} f](\xi_1, \xi_2, T).
\end{aligned}$$

□

Lemma 4.8. *We have*

$$A_{s,t,0} = \left(\prod_{n=1}^{|i-x|} \left[\begin{matrix} n \\ 1 \end{matrix} \right]_q \right)^{-1} \left(\prod_{n=1}^{|j-u|} \left[\begin{matrix} n \\ 1 \end{matrix} \right]_q \right)^{-1} (R_2)^{u-t} (R_1)^{x-s} (L_1)^{i-s} (L_2)^{j-t} |_{L(X_{i,j})}.$$

Proof. Let $f \in L(X_{i,j})$, $(\xi_1, \xi_2, T) \in X_{x,u}$. If $x \leq i$ and $u \leq j$, we have

$$\begin{aligned} [A_{s,t,0}f](\xi_1, \xi_2, T) &= \sum_{\substack{\xi_1 \subset \eta_1 \subset V_a \\ \dim \eta_1 = i}} \sum_{\substack{\xi_2 \subset \eta_2 \subset V_b \\ \dim \eta_2 = j}} \sum_{\substack{S \in B(\eta_2, V_a / \eta_1) \\ \pi(\eta_1)T = S|_{\xi_2}}} f(\eta_1, \eta_2, S) \\ &= \left(\prod_{n=1}^{j-u} \begin{bmatrix} n \\ 1 \end{bmatrix}_q \right)^{-1} \sum_{\substack{\xi_1 \subset \eta_1 \subset V_a \\ \dim \eta_1 = i}} \left[(L_2)^{j-u} f \right] (\eta_1, \xi_2, \pi(\eta_1)T) \\ &= \left(\prod_{n=1}^{i-x} \begin{bmatrix} n \\ 1 \end{bmatrix}_q \right)^{-1} \left(\prod_{n=1}^{j-u} \begin{bmatrix} n \\ 1 \end{bmatrix}_q \right)^{-1} \left[(L_1)^{i-x} (L_2)^{j-u} f \right] (\xi_1, \xi_2, T). \end{aligned}$$

If $x \leq i$ and $u > j$, we have

$$\begin{aligned} [A_{s,t,0}f](\xi_1, \xi_2, T) &= \sum_{\substack{\xi_1 \subset \eta_1 \subset V_a \\ \dim \eta_1 = i}} \sum_{\substack{\eta_2 \subset \xi_2 \\ \dim \eta_2 = j}} \sum_{\substack{S \in B(\eta_2, V_a / \eta_1) \\ \pi(\eta_1)T|_{\eta_2} = S}} f(\eta_1, \eta_2, S) \\ &= \sum_{\substack{\xi_1 \subset \eta_1 \subset V_a \\ \dim \eta_1 = i}} \sum_{\substack{\eta_2 \subset \xi_2 \\ \dim \eta_2 = j}} f(\eta_1, \eta_2, \pi(\eta_1)T|_{\eta_2}) \\ &= \left(\prod_{n=1}^{i-x} \begin{bmatrix} n \\ 1 \end{bmatrix}_q \right)^{-1} \sum_{\substack{\eta_2 \subset \xi_2 \\ \dim \eta_2 = j}} \left[(L_1)^{i-x} f \right] (\xi_1, \eta_2, T|_{\eta_2}) \\ &= \left(\prod_{n=1}^{i-x} \begin{bmatrix} n \\ 1 \end{bmatrix}_q \right)^{-1} \left(\prod_{n=1}^{u-j} \begin{bmatrix} n \\ 1 \end{bmatrix}_q \right)^{-1} \left[(R_2)^{u-j} (L_1)^{i-x} f \right] (\xi_1, \xi_2, T). \end{aligned}$$

If $x > i$ and $u \leq j$, we have

$$\begin{aligned} [A_{s,t,0}f](\xi_1, \xi_2, T) &= \sum_{\substack{\eta_1 \subset \xi_1 \\ \dim \eta_1 = i}} \sum_{\substack{\xi_2 \subset \eta_2 \subset V_b \\ \dim \eta_2 = j}} \sum_{\substack{S \in B(\eta_2, V_a / \eta_1) \\ T = \pi(\xi_1)S|_{\xi_2}}} f(\eta_1, \eta_2, S) \\ &= \sum_{\substack{\eta_1 \subset \xi_1 \\ \dim \eta_1 = i}} \sum_{\substack{\xi_2 \subset \eta_2 \subset V_b \\ \dim \eta_2 = j}} \sum_{\substack{S' \in B(\xi_2, V_a / \eta_1) \\ T = \pi(\xi_1)S'}} \sum_{\substack{S \in B(\eta_2, V_a / \eta_1) \\ S' = S|_{\xi_2}}} f(\eta_1, \eta_2, S) \\ &= \left(\prod_{n=1}^{j-u} \begin{bmatrix} n \\ 1 \end{bmatrix}_q \right)^{-1} \sum_{\substack{\eta_1 \subset \xi_1 \\ \dim \eta_1 = i}} \sum_{\substack{S' \in B(\xi_2, V_a / \eta_1) \\ T = \pi(\xi_1)S'}} \left[(L_2)^{j-u} f \right] (\eta_1, \xi_2, S') \\ &= \left(\prod_{n=1}^{x-i} \begin{bmatrix} n \\ 1 \end{bmatrix}_q \right)^{-1} \left(\prod_{n=1}^{j-u} \begin{bmatrix} n \\ 1 \end{bmatrix}_q \right)^{-1} \left[(R_1)^{x-i} (L_2)^{j-u} f \right] (\xi_1, \xi_2, T). \end{aligned}$$

If $x > i$ and $u > j$, we have

$$\begin{aligned} [A_{s,t,0}f](\xi_1, \xi_2, T) &= \sum_{\substack{\eta_1 \subset \xi_1 \\ \dim \eta_1 = i}} \sum_{\substack{\eta_2 \subset \xi_2 \\ \dim \eta_2 = j}} \sum_{\substack{S \in B(\eta_2, V_a / \eta_1) \\ T|_{\eta_2} = \pi(\xi_1)S}} f(\eta_1, \eta_2, S) \\ &= \left(\prod_{n=1}^{x-i} \begin{bmatrix} n \\ 1 \end{bmatrix}_q \right)^{-1} \sum_{\substack{\eta_2 \subset \xi_2 \\ \dim \eta_2 = j}} \left[(R_1)^{x-i} f \right] (\xi_1, \eta_2, T|_{\eta_2}) \\ &= \left(\prod_{n=1}^{x-i} \begin{bmatrix} n \\ 1 \end{bmatrix}_q \right)^{-1} \left(\prod_{n=1}^{u-j} \begin{bmatrix} n \\ 1 \end{bmatrix}_q \right)^{-1} \left[(R_2)^{u-j} (R_1)^{x-i} f \right] (\xi_1, \xi_2, T). \end{aligned}$$

□

Lemma 4.9. Let $\pi_{m,n,r} = \pi_{m,n,r}^{x,u,i,j} : V_{m,n,r}^{i,j} \rightarrow V_{m,n,r}^{x,u}$ be the fixed isomorphism maps. Then we have

$$\begin{aligned} & (R_2)^{u-t} (R_1)^{x-s} (L_1)^{i-s} (L_2)^{j-t} |_{V_{m,n,r}^{i,j}} \\ &= \sum_{(m,n,r) \in L} \left(\prod_{r_2=0}^{u-t-1} \frac{q^{t-n-r+1+r_2} - 1}{q-1} \right) \left(\prod_{r_1=0}^{x-s-1} q^t \frac{q^{s-m+1+r_1} - 1}{q-1} \right) \\ &\quad \times \left(\prod_{l_1=0}^{i-s-1} \frac{q^{a-r-i+1+l_1} - q^m}{q-1} \right) \left(\prod_{l_2=0}^{j-t-1} q^{a-i} \frac{q^{b-j+1+l_2} - q^n}{q-1} \right) \pi_{m,n,r}. \end{aligned}$$

Proof. Dunkl [8, Theorem 4.17] have proved that

$$\begin{aligned} L_1|_{V_{m,n,r}^{i,j}} &= \frac{q^{a-r-i+1} - q^m}{q-1} \pi_{m,n,r}^{i-1,j,i,j}, \\ L_2|_{V_{m,n,r}^{i,j}} &= q^{a-i} \frac{q^{b-j+1} - q^n}{q-1} \pi_{m,n,r}^{i,j-1,i,j}, \\ R_1|_{V_{m,n,r}^{i,j}} &= q^j \frac{q^{i-m+1} - 1}{q-1} \pi_{m,n,r}^{i+1,j,i,j}, \\ R_2|_{V_{m,n,r}^{i,j}} &= \frac{q^{j-n-r+1} - 1}{q-1} \pi_{m,n,r}^{i,j+1,i,j}. \end{aligned}$$

From these formulas, we obtain the result. \square

Lemma 4.10.

$$\begin{aligned} \left(\prod_{n_2=1}^{|i-x|} \begin{bmatrix} n_2 \\ 1 \end{bmatrix}_q \right) &= \left(\prod_{r_1=0}^{x-s-1} \frac{q^{r_1+1} - 1}{q-1} \right) \left(\prod_{l_1=0}^{i-s-1} \frac{q^{l_1+1} - 1}{q-1} \right). \\ \left(\prod_{n_1=1}^{|j-u|} \begin{bmatrix} n_1 \\ 1 \end{bmatrix}_q \right) &= \left(\prod_{r_2=0}^{u-t-1} \frac{q^{r_2+1} - 1}{q-1} \right) \left(\prod_{l_2=0}^{j-t-1} \frac{q^{l_2+1} - 1}{q-1} \right). \end{aligned}$$

Proof. This is by the definition of the q -binomial coefficient. \square

From these lemmas above, we obtain the value of $P_{(m,n,r),(s,t,0)}^{x,u,i,j}$.

Theorem 4.11. $P_{(m,n,r),(s,t,0)}^{x,u,i,j} =$

$$q^{t(x-s)} q^{m(i-s)} q^{(a-i+n)(j-t)} \begin{bmatrix} u-n-r \\ u-t \end{bmatrix}_q \begin{bmatrix} x-m \\ x-s \end{bmatrix}_q \begin{bmatrix} b-t-n \\ j-t \end{bmatrix}_q \begin{bmatrix} a-r-s-m \\ i-s \end{bmatrix}_q.$$

Proof. By Lemma 4.8, Lemma 4.9 and Lemma 4.10, we have

$$\begin{aligned} A_{s,t,0} &= \sum_{(m,n,r) \in L} \left(\prod_{r_2=0}^{u-t-1} \frac{q^{t-n-r+1+r_2} - 1}{q^{r_2+1} - 1} \right) \left(\prod_{r_1=0}^{x-s-1} q^t \frac{q^{s-m+1+r_1} - 1}{q^{r_1+1} - 1} \right) \\ &\quad \times \left(\prod_{l_1=0}^{i-s-1} \frac{q^{a-r-i+1+l_1} - q^m}{q^{l_1+1} - 1} \right) \left(\prod_{l_2=0}^{j-t-1} q^{a-i} \frac{q^{b-j+1+l_2} - q^n}{q^{l_2+1} - 1} \right) \pi_{m,n,r} \\ &= \sum_{(m,n,r) \in L} \left(\prod_{r_2=0}^{u-t-1} \frac{q^{t-n-r+1+r_2} - 1}{q^{r_2+1} - 1} \right) \left(\prod_{r_1=0}^{x-s-1} q^t \frac{q^{s-m+1+r_1} - 1}{q^{r_1+1} - 1} \right) \\ &\quad \times \left(\prod_{l_1=0}^{i-s-1} q^m \frac{q^{a-r-i+1+l_1-m} - 1}{q^{l_1+1} - 1} \right) \left(\prod_{l_2=0}^{j-t-1} q^{a-i+n} \frac{q^{b-j+1+l_2-n} - 1}{q^{l_2+1} - 1} \right) \pi_{m,n,r} \\ &= \sum_{(m,n,r) \in L} \begin{bmatrix} u-n-r \\ u-t \end{bmatrix}_q q^{t(x-s)} \begin{bmatrix} x-m \\ x-s \end{bmatrix}_q q^{m(i-s)} \end{aligned}$$

$$\begin{aligned}
& \times \begin{bmatrix} a-r-s-m \\ i-s \end{bmatrix}_q q^{(a-i+n)(j-t)} \begin{bmatrix} b-t-n \\ j-t \end{bmatrix}_q \pi_{m,n,r} \\
= & \sum_{(m,n,r) \in L} q^{t(x-s)} q^{m(i-s)} q^{(a-i+n)(j-t)} \begin{bmatrix} u-n-r \\ u-t \end{bmatrix}_q \begin{bmatrix} x-m \\ x-s \end{bmatrix}_q \\
& \times \begin{bmatrix} b-t-n \\ j-t \end{bmatrix}_q \begin{bmatrix} a-r-s-m \\ i-s \end{bmatrix}_q \pi_{m,n,r}.
\end{aligned}$$

□

Next we calculate the value $|R_{v,w,z}|$ for any $(v, w, z) \in K$.

Lemma 4.12. *For any vector spaces K_1, K_2 over \mathbb{F}_q such that $\dim K_1 = \kappa_1, \dim K_2 = \kappa_2$, the number of $S \in B(K_1, K_2)$ with $\text{rank } S = r$ ($0 \leq r \leq \min\{\kappa_1, \kappa_2\}$) is*

$$\frac{(q^{\kappa_1})_r (q^{\kappa_2})_r}{(q^{-1})_r} q^{-r}.$$

Proof. First we determine the choices of the kernel of S ;

$$|\{(\kappa_1 - r)\text{-dimensional subspace in } K_1\}| = |\{r\text{-dimensional subspace in } K_1\}| = \begin{bmatrix} \kappa_1 \\ r \end{bmatrix}_q.$$

Next, for given $\text{Ker } S$, the linear map S is determined by the induced injective linear map $\bar{S} : K_1 / \text{Ker } S \rightarrow K_2$. The number of choices of \bar{S} is equal to that of the r -tuples of linearly independent vectors in K_2 , i.e.,

$$(q^{\kappa_2} - 1)(q^{\kappa_2} - q) \cdots (q^{\kappa_2} - q^{r-1}).$$

Thus the number we want to determine is

$$\begin{aligned}
& \begin{bmatrix} \kappa_1 \\ r \end{bmatrix}_q (q^{\kappa_2} - 1)(q^{\kappa_2} - q) \cdots (q^{\kappa_2} - q^{r-1}) \\
= & \frac{(1 - q^{\kappa_1})(1 - q^{\kappa_1-1}) \cdots (1 - q^{\kappa_1-r+1})}{(1 - q^r)(1 - q^{r-1}) \cdots (1 - q)} (q^{\kappa_2} - 1)(q^{\kappa_2} - q) \cdots (q^{\kappa_2} - q^{r-1}) \\
= & (q^{\kappa_1})_r (q^{\kappa_2})_r \frac{(-1)^r q^{r(r-1)/2}}{(1 - q^r)(1 - q^{r-1}) \cdots (1 - q)} \\
= & (q^{\kappa_1})_r (q^{\kappa_2})_r \frac{q^{-r}}{(q^{-1})_r}.
\end{aligned}$$

□

Theorem 4.13. $|R_{v,w,z}| = q^{(a-x)u} q^{(x-v)(i-v+w)} q^{(a-i+u-w)(j-w)}$

$$\times \begin{bmatrix} a \\ x \end{bmatrix}_q \begin{bmatrix} x \\ v \end{bmatrix}_q \begin{bmatrix} a-x \\ i-v \end{bmatrix}_q \begin{bmatrix} b \\ u \end{bmatrix}_q \begin{bmatrix} u \\ w \end{bmatrix}_q \begin{bmatrix} b-u \\ j-w \end{bmatrix}_q \frac{(q^{a-x-i+v})_z (q^w)_z}{(q^{-1})_z} q^{-z}.$$

Proof.

$$\begin{aligned}
& |R_{v,w,z}| \\
= & \sum_{\substack{\dim \xi_1 = x \\ \dim \eta_1 = i \\ \dim (\xi_1 \cap \eta_1) = v}} \sum_{\substack{\dim \xi_2 = u \\ \dim \eta_2 = j \\ \dim (\xi_2 \cap \eta_2) = w}} \left| \left\{ (S, S') \mid \begin{array}{c} S \in B(\xi_2, V_a / \xi_1), \quad S' \in B(\eta_2, V_a / \eta_1) \\ \text{rank}(\pi(\xi_1 + \eta_1) S|_{\xi_2 \cap \eta_2}) = z \\ -\pi(\xi_1 + \eta_1) S'|_{\xi_2 \cap \eta_2} = z \end{array} \right\} \right| \\
= & \sum_{\substack{\dim \xi_1 = x \\ \dim \eta_1 = i \\ \dim (\xi_1 \cap \eta_1) = v}} \sum_{\substack{\dim \xi_2 = u \\ \dim \eta_2 = j \\ \dim (\xi_2 \cap \eta_2) = w}} q^{(a-x)(u-w)} q^{(a-i)(j-w)}
\end{aligned}$$

$$\begin{aligned}
& \times \left| \left\{ (S_0, S'_0) \mid \begin{array}{l} S_0 \in B(\xi_2 \cap \eta_2, V_a / \xi_1), \quad S'_0 \in B(\xi_2 \cap \eta_2, V_a / \eta_1) \\ \text{rank}(\pi(\xi_1 + \eta_1)S_0 - \pi(\xi_1 + \eta_1)S'_0) = z \end{array} \right\} \right| \\
&= q^{(a-x)(u-w)} q^{(a-i)(j-w)} \\
&\quad \times \sum_{\substack{\dim \xi_1 = x \\ \dim \eta_1 = i \\ \dim (\xi_1 \cap \eta_1) = v}} \sum_{\substack{\dim \xi_2 = u \\ \dim \eta_2 = j \\ \dim (\xi_2 \cap \eta_2) = w}} q^{(i-v)w} q^{(x-v)w} \left| \left\{ (S_1, S'_1) \mid \begin{array}{l} S_1, S'_1 \in B(\xi_2 \cap \eta_2, V_a / (\xi_1 + \eta_1)) \\ \text{rank}(S_1 - S'_1) = z \end{array} \right\} \right| \\
&= q^{(a-x)(u-w)} q^{(a-i)(j-w)} q^{(i-v)w} q^{(x-v)w} \\
&\quad \times \sum_{\substack{\dim \xi_1 = x \\ \dim \eta_1 = i \\ \dim (\xi_1 \cap \eta_1) = v}} \sum_{\substack{\dim \xi_2 = u \\ \dim \eta_2 = j \\ \dim (\xi_2 \cap \eta_2) = w}} \left| \left\{ (S_1, T_1) \mid \begin{array}{l} S_1, T_1 \in B(\xi_2 \cap \eta_2, K^a / (\xi_1 + \eta_1)) \\ \text{rank}(T_1) = z \end{array} \right\} \right| \\
&= q^{(a-x)(u-w)} q^{(a-i)(j-w)} q^{(i-v)w} q^{(x-v)w} \\
&\quad \times \sum_{\substack{\dim \xi_1 = x \\ \dim \eta_1 = i \\ \dim (\xi_1 \cap \eta_1) = v}} \sum_{\substack{\dim \xi_2 = u \\ \dim \eta_2 = j \\ \dim (\xi_2 \cap \eta_2) = w}} q^{w(a-x-i+v)} \frac{(q^{a-x-i+v})_z (q^w)_z}{(q^{-1})_z} q^{-z} \quad \text{by Lemma 4.12} \\
&= q^{(a-x)(u-w)} q^{(a-i)(j-w)} q^{(i-v)w} q^{(x-v)w} \\
&\quad \times \begin{bmatrix} a \\ x \end{bmatrix}_q \begin{bmatrix} x \\ v \end{bmatrix}_q \begin{bmatrix} a-x \\ i-v \end{bmatrix}_q q^{(x-v)(i-v)} \begin{bmatrix} b \\ u \end{bmatrix}_q \begin{bmatrix} u \\ w \end{bmatrix}_q \begin{bmatrix} b-u \\ j-w \end{bmatrix}_q q^{(u-w)(j-w)} \\
&\quad \times q^{w(a-x-i+v)} \frac{(q^{a-x-i+v})_z (q^w)_z}{(q^{-1})_z} q^{-z} \\
&= q^{(a-x)u} q^{(x-v)(i-v+w)} q^{(a-i+u-w)(j-w)} \\
&\quad \times \begin{bmatrix} a \\ x \end{bmatrix}_q \begin{bmatrix} x \\ v \end{bmatrix}_q \begin{bmatrix} a-x \\ i-v \end{bmatrix}_q \begin{bmatrix} b \\ u \end{bmatrix}_q \begin{bmatrix} u \\ w \end{bmatrix}_q \begin{bmatrix} b-u \\ j-w \end{bmatrix}_q \frac{(q^{a-x-i+v})_z (q^w)_z}{(q^{-1})_z} q^{-z}.
\end{aligned}$$

□

Theorem 4.14. $|R_{s,t,0}| =$

$$q^{(a-x)u} q^{(x-s)(i-s+t)} q^{(a-i+u-t)(j-t)} \begin{bmatrix} a \\ s \end{bmatrix}_q \begin{bmatrix} a-s \\ x-s \end{bmatrix}_q \begin{bmatrix} a-x \\ i-s \end{bmatrix}_q \begin{bmatrix} b \\ t \end{bmatrix}_q \begin{bmatrix} b-t \\ u-t \end{bmatrix}_q \begin{bmatrix} b-u \\ j-t \end{bmatrix}_q.$$

Proof. As a corollary of the previous theorem. □

Finally we have determined the values $\mathcal{P}^{x,u,i,j}$ and $\mathcal{Q}^{x,u,i,j}$ and finish this subsection.

4.2.3 Values of $P_{(m,n,r),(v,w,z)}^{x,u,i,j}$ with $x = i$ and $u = j$

In this subsection we list some formulas to determine the values of $P_{(m,n,r),(v,w,z)}^{x,u,i,j}$ under the conditions $x = i$ and $u = j$, which were obtained as corollary of the previous subsection. The contents in this subsection are only for the calculations later.

Lemma 4.15. Suppose $x = i (= s)$ and $u = j (= t)$. Then we have

$$P_{(m,n,r),(s,t,0)}^{x,u,i,j} = 1.$$

Proof. By Theorem 4.11. □

Lemma 4.16. Suppose $x = i (= s)$ and $u = j (= t)$. Then we have

$$\frac{|R_{v,w,z}|}{|R_{s,t,0}|} = q^{(s-v)(s-v+w)} q^{(a-s+t-w)(t-w)} \begin{bmatrix} s \\ v \end{bmatrix}_q \begin{bmatrix} a-s \\ s-v \end{bmatrix}_q \begin{bmatrix} t \\ w \end{bmatrix}_q \begin{bmatrix} b-t \\ t-w \end{bmatrix}_q \frac{(q^{a-2s+v})_z (q^w)_z}{(q^{-1})_z} q^{-z}.$$

In addition, we have the following:

(i) If $(v, w, z) = (s, t, 1)$,

$$\frac{|R_{v,w,z}|}{|R_{s,t,0}|} = \frac{(1 - q^{a-s})(1 - q^t)}{(q - 1)}.$$

(ii) If $(v, w, z) = (s - 1, t, 0)$,

$$\frac{|R_{v,w,z}|}{|R_{s,t,0}|} = q^{t+1} \frac{(1 - q^s)(1 - q^{a-s})}{(1 - q)^2}.$$

(iii) If $(v, w, z) = (s, t - 1, 0)$,

$$\frac{|R_{v,w,z}|}{|R_{s,t,0}|} = q^{a-s+1} \frac{(1 - q^t)(1 - q^{b-t})}{(1 - q)^2}.$$

Proof. Using Theorem 4.14, we have

$$|R_{s,t,0}| = q^{(a-s)t} \begin{bmatrix} a \\ s \end{bmatrix}_q \begin{bmatrix} b \\ t \end{bmatrix}_q.$$

Using Theorem 4.13, we have $|R_{v,w,z}| =$

$$q^{(a-s)t} q^{(s-v)(s-v+w)} q^{(a-s+t-w)(t-w)} \begin{bmatrix} a \\ s \end{bmatrix}_q \begin{bmatrix} s \\ v \end{bmatrix}_q \begin{bmatrix} a-s \\ s-v \end{bmatrix}_q \begin{bmatrix} b \\ t \end{bmatrix}_q \begin{bmatrix} t \\ w \end{bmatrix}_q \begin{bmatrix} b-t \\ t-w \end{bmatrix}_q \frac{(q^{a-2s+v})_z (q^w)_z}{(q^{-1})_z} q^{-z}.$$

These two equations imply the formula we want to prove. In addition, if we apply $(v, w, z) = (s, t, 1), (s - 1, t, 0), (s, t - 1, 0)$, we obtain the results. \square

Lemma 4.17. For q -Krawtchouk polynomials and q -Hahn polynomials, we have the following:

(i)

$$\frac{K_m(x; a, b; q)}{K_m(0; a, b; q)} = \sum_{j=0}^m \frac{(q^{-m}; q)_j (q^{-x}; q)_j q^j}{(q^{-a}; q)_j (q^{-b}; q)_j (q; q)_j}.$$

(ii)

$$\frac{E_m(b, a, c, c-x; q^{-1})}{E_m(b, a, c, c; q^{-1})} = \frac{E_m(a, b, c, x; q)}{E_m(a, b, c, 0; q)} = \sum_{j=0}^m \frac{(q^{-m}; q)_j (q^{-a-b+m-1}; q)_j (q^{-x}; q)_j q^j}{(q^{-a}; q)_j (q^{-c}; q)_j (q; q)_j}.$$

(iii)

$$\frac{K_m(0; a, b; q)}{K_m(0; c, d; q)} = \frac{(q^a)_m (q^b)_m}{(q^c)_m (q^d)_m}.$$

Proof. (i) This is from the definition of q -Krawtchouk polynomials.

(ii) We use the formula [8, Proposition 2.3]:

$$E_m(b, a, c, c-x; q^{-1}) = (-1)^m q^{m(a+b+c)-m(m-1)/2} E_m(a, b, c, x; q).$$

(iii) This comes from the definition of q -Krawtchouk polynomials. \square

Finally we construct a simple formula for the case $x = i$ and $u = j$.

Theorem 4.18. Suppose $x = i (= s)$ and $u = j (= t)$. Then we have the following:

(i) If $(v, w, z) = (s, t, 1)$,

$$P_{(m,n,r),(v,w,z)}^{x,u,i,j} = \frac{1}{(q-1)} (1 - q^{a-s} - q^t + q^{a-r-s+t}).$$

(ii) If $(v, w, z) = (s - 1, t, 0)$,

$$P_{(m,n,r),(v,w,z)}^{x,u,i,j} = \frac{q^{t+1}}{(1-q)^2} \left(1 - q^{a-r-s} - q^s + q^{a-r-m} + q^{m-1} - q^{-1} \right).$$

(iii) If $(v, w, z) = (s, t - 1, 0)$,

$$P_{(m,n,r),(v,w,z)}^{x,u,i,j} = \frac{q^{a-s+1}}{(1-q)^2} \left(1 - q^{b-t} - q^{t-r} + q^{b-n-r} + q^{n-1} - q^{-1} \right).$$

Proof. (i) Suppose $(v, w, z) = (s, t, 1)$.

$$\begin{aligned} & P_{(m,n,r),(v,w,z)}^{x,u,i,j} \\ &= \frac{|R_{v,w,z}|}{|R_{s,t,0}|} P_{(m,n,r),(s,t,0)}^{x,u,i,j} \omega_{m,n,r}(v, w, z) && \text{by (4.25)} \\ &= \frac{(1 - q^{a-s})(1 - q^t)}{(q - 1)} \omega_{m,n,r}(v, w, z) && \text{by Lemmas 4.15 and 4.16} \\ &= \frac{(1 - q^{a-s})(1 - q^t)}{(q - 1)} \frac{K_r(1; t, a - s; q)}{K_r(0; t, a - s; q)} \\ &= \frac{(1 - q^{a-s})(1 - q^t)}{(q - 1)} \sum_{j=0}^r \frac{(q^{-r}; q)_j (q^{-1}; q)_j q^j}{(q^{-t}; q)_j (q^{-a+s}; q)_j (q; q)_j} && \text{by Lemma 4.17} \\ &= \frac{(1 - q^{a-s})(1 - q^t)}{(q - 1)} \left\{ 1 + \frac{(1 - q^{-r})(1 - q^{-1})q}{(1 - q^{-t})(1 - q^{-a+s})(1 - q)} \right\} \\ &= \frac{1}{(q - 1)} (1 - q^{a-s} - q^t + q^{a-r-s+t}). \end{aligned}$$

(ii) Suppose $(v, w, z) = (s - 1, t, 0)$.

$$\begin{aligned} & P_{(m,n,r),(v,w,z)}^{x,u,i,j} \\ &= \frac{|R_{v,w,z}|}{|R_{s,t,0}|} P_{(m,n,r),(s,t,0)}^{x,u,i,j} \omega_{m,n,r}(v, w, z) && \text{by (4.25)} \\ &= q^{t+1} \frac{(1 - q^s)(1 - q^{a-s})}{(1 - q)^2} \omega_{m,n,r}(v, w, z) && \text{by Lemmas 4.15 and 4.16} \\ &= q^{t+1} \frac{(1 - q^s)(1 - q^{a-s})}{(1 - q)^2} \\ &\quad \times \frac{E_m(s, a - r - s, s, s - 1; q^{-1})}{E_m(s, a - r - s, s, s; q^{-1})} \frac{K_r(0; t, a - s - 1; q)}{K_r(0; t, a - s; q)} \\ &= q^{t+1} \frac{(1 - q^s)(1 - q^{a-s})}{(1 - q)^2} \\ &\quad \times \left(\sum_{j=0}^m \frac{(q^{-m}; q)_j (q^{-a+r+m-1}; q)_j (q^{-1}; q)_j q^j}{(q^{-a+r+s}; q)_j (q^{-s}; q)_j (q; q)_j} \right) \frac{(q^{a-s-1})_r}{(q^{a-s})_r} && \text{by Lemma 4.17} \\ &= q^{t+1} \frac{(1 - q^s)(1 - q^{a-s})}{(1 - q)^2} \\ &\quad \times \left\{ 1 + \frac{(1 - q^{-m})(1 - q^{-a+r+m-1})(1 - q^{-1})q}{(1 - q^{-a+r+s})(1 - q^{-s})(1 - q)} \right\} \frac{1 - q^{a-s-r}}{1 - q^{a-s}} \\ &= \frac{q^{t+1}}{(1 - q)^2} (1 - q^{a-r-s} - q^s + q^{a-r-m} + q^{m-1} - q^{-1}). \end{aligned}$$

(iii) Suppose $(v, w, z) = (s, t - 1, 0)$.

$$P_{(m,n,r),(v,w,z)}^{x,u,i,j}$$

$$\begin{aligned}
&= \frac{|R_{v,w,z}|}{|R_{s,t,0}|} P_{(m,n,r),(s,t,0)}^{x,u,i,j} \omega_{m,n,r}(v, w, z) && \text{by (4.25)} \\
&= q^{a-s+1} \frac{(1-q^t)(1-q^{b-t})}{(1-q)^2} \omega_{m,n,r}(v, w, z) && \text{by Lemmas 4.15 and 4.16} \\
&= q^{a-s+1} \frac{(1-q^t)(1-q^{b-t})}{(1-q)^2} \\
&\quad \times \frac{E_n(t-r, b-t, t-r, t-r-1; q^{-1})}{E_n(t-r, b-t, t-r, t-r; q^{-1})} \frac{K_r(0; t-1, a-s; q)}{K_r(0; t, a-s; q)} \\
&= q^{a-s+1} \frac{(1-q^t)(1-q^{b-t})}{(1-q)^2} \\
&\quad \times \left(\sum_{j=0}^n \frac{(q^{-n}; q)_j (q^{-b+r+n-1}; q)_j (q^{-1}; q)_j}{(q^{-b+t}; q)_j (q^{-t+r}; q)_j (q; q)_j} q^j \right) \frac{(q^{t-1})_r}{(q^t)_r} && \text{by Lemma 4.17} \\
&= q^{a-s+1} \frac{(1-q^t)(1-q^{b-t})}{(1-q)^2} \\
&\quad \times \left\{ 1 + \frac{(1-q^{-n})(1-q^{-b+r+n-1})(1-q^{-1})q}{(1-q^{-b+t})(1-q^{-t+r})(1-q)} \right\} \frac{1-q^{t-r}}{1-q^t} \\
&= \frac{q^{a-s+1}}{(1-q)^2} (1 - q^{b-t} - q^{t-r} + q^{b-n-r} + q^{n-1} - q^{-1}).
\end{aligned}$$

□

4.3 The Terwilliger Algebras of the Grassmann Graphs

We continue to use the same symbols from the last section. We denote by X the set of all subspaces of a vector space V over a finite field \mathbb{F}_q , which has the decomposition $X = \sqcup_{i,j} X_{i,j}$. We also let the dimension of V be $a+b$.

We start with the definition of Grassmann graphs. For a nonnegative integer D with $a \leq D \leq a+b$, let X_D be the set of all subspaces of V of dimension D :

$$X_D = \bigsqcup_{\max\{0, D-b\} \leq i \leq a} X_{i,D-i}. \quad (4.31)$$

The Grassmann graph is the graph with vertex set X_D where two subspaces $\xi, \eta \in X_D$ are adjacent if $\dim(\xi \cap \eta) = D-1$. We denote the Grassmann graph by $J_q(a+b, D)$. The Grassmann graph is distance-regular, that is for any vertices $\eta, \zeta \in X_D$ the number $|\{\xi \in X_D \mid \partial(\xi, \eta) = i, \partial(\xi, \zeta) = j\}|$ depends only on i, j and $\partial(\eta, \zeta)$. Here ∂ denotes the graph distance. Define the distance-relations by

$$R_k = \{(\xi, \eta) \in X_D \times X_D \mid \partial(\xi, \eta) = k\}, \quad (0 \leq k \leq D). \quad (4.32)$$

Then $(X_D, \{R_k\}_{k=0}^D)$ is a symmetric association scheme. More generally, any distance-regular graph forms a symmetric association scheme by its distance-relations. Let \mathcal{A} be the Bose-Mesner algebra of the symmetric association scheme obtained from the Grassmann graph $J_q(a+b, D)$. Then it is known that \mathcal{A} is generated only by the adjacency matrix A of the Grassmann graph.

For convenience, we use the symbols Y_i ($\max\{0, D-b\} \leq i \leq a$) for the decomposition of X_D :

$$Y_i := X_{i,D-i} = \{\xi \in X_D \mid \dim(\xi \cap V_a) = a-i\}. \quad (4.33)$$

Remark that when $D \leq (a+b)/2$, the subset Y_0 of X_D is an important object called a descendent of the Grassmann graph $J_q(a+b, D)$. See Brouwer–Godsil–Koolen–Martin [3] and Tanaka [23, 24] for detailed information about descendants.

Define the square matrices E_i^* ($\max\{0, D-b\} \leq i \leq a$) indexed by X_D : for $\xi, \eta \in X_D$,

$$(E_i^*)_{\xi, \eta} = \begin{cases} 1 & \text{if } \xi = \eta \in Y_i, \\ 0 & \text{otherwise.} \end{cases} \quad (4.34)$$

The matrices E_i^* are diagonal and called the dual idempotents.

Let $\text{Mat}_{X_D}(\mathbb{C})$ denote the complex matrix algebra whose rows and columns are indexed by X_D . Define the algebra \mathcal{T} by

$$\mathcal{T} = \mathbb{C}[A, E_{\max\{0, D-b\}}^*, \dots, E_a^*] \subset \text{Mat}_{X_D}(\mathbb{C}), \quad (4.35)$$

where A denotes the adjacency matrix of the Grassmann graph $J_q(a+b, D)$. When $D \leq (a+b)/2$, the algebra \mathcal{T} is called the Terwilliger algebra with respect to the subset Y_0 of X_D .

Note that Terwilliger algebras were first introduced by Terwilliger [26], and were extended to the general definition by Suzuki [22]. In this paper, we espouse the definition by H. Suzuki. Remark that if $a = D$, that is Y_0 contains only one vertex V_a , the algebra \mathcal{T} is exactly the same as the original Terwilliger algebra defined by P. Terwilliger.

In this section, we determine all irreducible representations of \mathcal{T} . Here we define a new index set L' from the previous sets (4.18);

$$L' = \bigcup_{\max\{0, D-b\} \leq i \leq a} L_{(i, D-i)}. \quad (4.36)$$

In the previous section, we have essentially considered the space $\text{Hom}_G(L(G/H_{i,j}), L(G/H_{x,u}))$ (cf. Chapter 2). Here we only use the case $j = D - i$ and $u = D - x$ for suitable i and x . Let $\lambda : G \rightarrow \text{GL}(L(X_D))$ be the permutation representation of G on $L(X_D)$ and $\lambda_i : G \rightarrow \text{GL}(L(Y_i))$ be the permutation representation on $L(Y_i)$, which is a subrepresentation of λ . Theorem 4.6 (iii) says that we have the decomposition

$$\lambda_i = \bigoplus_{(m,n,r) \in L_{(i, D-i)}} \lambda_{m,n,r}, \quad (4.37)$$

such that $\lambda_{m,n,r} : G \rightarrow \text{GL}(V_{m,n,r})$ are irreducible. Setting the number $d(m, n, r) = |\{i \mid \lambda_{m,n,r} < \lambda_i\}| - 1$, we have

$$\text{Hom}_G(L(X_D), L(X_D)) \simeq \bigoplus_{(m,n,r) \in L'} \text{Mat}_{d(m,n,r)+1}(\mathbb{C}), \quad (4.38)$$

where $\text{Mat}_k(\mathbb{C})$ denotes the set of all $k \times k$ matrices whose entries are in \mathbb{C} . Here we used Schur's lemma. For $(m, n, r) \in L'$, let $\varphi_{m,n,r} : \text{Hom}_G(L(X_D), L(X_D)) \rightarrow \text{Mat}_{d(m,n,r)+1}(\mathbb{C})$ be the irreducible representation obtained above. We remark that the matrices A and E_i^* commute with the G -action and that therefore \mathcal{T} is a subalgebra of $\text{Hom}_G(L(X_D), L(X_D))$. We see the $\varphi_{m,n,r}$ as representations of \mathcal{T} by restriction.

Then we determine the explicit values of $\varphi_{m,n,r}(E_i^*)$ and $\varphi_{m,n,r}(A)$.

Theorem 4.19. *For any $(m, n, r) \in L'$ and $\max\{0, D-b\} \leq i \leq a$, the matrix $\varphi_{m,n,r}(E_i^*)$ is diagonal: There exists $v = v(m, n, r)$ such that*

$$\varphi_{m,n,r}(E_i^*) = \begin{cases} \text{diag}(0, \dots, 0, \overset{i-v}{\underset{\vee}{1}}, 0, \dots, 0) & \text{if } v \leq i \leq v+d, \\ 0 & \text{otherwise,} \end{cases}$$

where $v = \max\{m, D-b+n\}$ and $v+d = \min\{a-r-m, D-n-r\}$

Proof. This is clear from the definition of the dual idempotents. \square

Theorem 4.20. *For any $(m, n, r) \in L'$, the matrix $\varphi_{m,n,r}(A)$ is tridiagonal:*

$$(\varphi_{m,n,r}(A))_{x,i} = \begin{cases} \frac{q^{a+b-m+1}}{(1-q)^2} (1 - q^{m-x})(1 - q^{D-b+n-x}) & \text{if } i = x-1, \\ \frac{1}{(1-q)^2} (q - 1 - q^{D+a-r-2x} - q^{D+1} \\ + q^{D+a-m-r-x+1} + q^{D+m-x} - q^{-D+a+b+1} \\ - q^{D+a-r-2x+1} + q^{a+b-n-r-x+1} + q^{a+n-x}) & \text{if } i = x, \\ \frac{q^m}{(1-q)^2} (1 - q^{D-n-r-x})(1 - q^{a-m-r-x}) & \text{if } i = x+1, \\ 0 & \text{otherwise,} \end{cases}$$

where the range of the index is $\max\{m, D-b+n\} \leq x, i \leq \min\{a-r-m, D-n-r\}$.

Proof. Here we write $K_{x,i}$ for the index set defined in (4.9) with $j = D - i$ and $u = D - x$. Let $A_{v,w,z}^{x,i}$ be “the adjacency matrix” of $\mathfrak{X}_{x,D-x,i,D-j}$ defined in (4.11). By the definition of the adjacency matrix A and Theorem 4.4, we have for $\max\{0, D-b\} \leq x, i \leq a$,

$$A_{\xi,\eta} = \sum_{\substack{(v,w,z) \in K_{x,i} \\ v+w-z=D-1}} \left(A_{v,w,z}^{x,i} \right)_{\xi,\eta} \quad (\xi \in Y_x, \eta \in Y_i).$$

Let $\left(P_{(m,n,r),(v,w,z)}^{x,u,i,j} \right)$ be “the first eigenmatrix” defined in (4.24) and determined in (4.25). Then we have

$$(\varphi_{m,n,r}(A))_{x,i} = \sum_{\substack{(v,w,z) \in K_{x,i} \\ v+w-z=D-1}} P_{(m,n,r),(v,w,z)}^{x,D-x,i,D-i}.$$

Note that the right-hand side will be undefined if (and only if) the indices x, i are out of the range. Recall that we have set $s = \min\{i, x\}$ and $t = \min\{j, u\}$.

(i) If $i \leq x - 2$ or $i \geq x + 2$, the right-hand side will be zero because the condition $v + w - z = D - 1$ never holds.

(ii) Let $i = x - 1$.

$$\begin{aligned} (\varphi_{m,n,r}(A))_{x,x-1} &= \sum_{\substack{(v,w,z) \in K_{x,x-1} \\ v+w-z=D-1}} P_{(m,n,r),(v,w,z)}^{x,D-x,x-1,D-x+1} \\ &= P_{(m,n,r),(x-1,D-x,0)}^{x,D-x,x-1,D-x+1} \\ &= P_{(m,n,r),(s,t,0)}^{x,D-x,x-1,D-x+1} \quad s = x - 1, t = D - x \\ &= q^{D-x} q^{a-x+1+n} \begin{bmatrix} x-m \\ 1 \end{bmatrix}_q \begin{bmatrix} b-D+x-n \\ 1 \end{bmatrix}_q \quad \text{by Theorem 4.11} \\ &= \frac{q^{a+b-m+1}}{(1-q)^2} (1 - q^{m-x})(1 - q^{D-b+n-x}). \end{aligned}$$

(iii) Let $i = x + 1$.

$$\begin{aligned} (\varphi_{m,n,r}(A))_{x,x+1} &= \sum_{\substack{(v,w,z) \in K_{x,x+1} \\ v+w-z=D-1}} P_{(m,n,r),(v,w,z)}^{x,D-x,x+1,D-x-1} \\ &= P_{(m,n,r),(x,D-x-1,0)}^{x,D-x,x+1,D-x-1} \\ &= P_{(m,n,r),(s,t,0)}^{x,D-x,x+1,D-x-1} \quad s = x - 1, t = D - x \\ &= q^m \begin{bmatrix} D-x-n-r \\ 1 \end{bmatrix}_q \begin{bmatrix} a-r-x-m \\ 1 \end{bmatrix}_q \quad \text{by Theorem 4.11} \\ &= \frac{q^m}{(1-q)^2} (1 - q^{D-n-r-x})(1 - q^{a-m-r-x}). \end{aligned}$$

(iv) Let $i = x$. Then we have

$$\begin{aligned} (\varphi_{m,n,r}(A))_{x,x} &= P_{(m,n,r),(x,D-x,1)}^{x,D-x,x,D-x} + P_{(m,n,r),(x-1,D-x,0)}^{x,D-x,x,D-x} + P_{(m,n,r),(x,D-x-1,0)}^{x,D-x,x,D-x} \\ &= P_{(m,n,r),(s,t,1)}^{x,D-x,x,D-x} + P_{(m,n,r),(s-1,t,0)}^{x,D-x,x,D-x} + P_{(m,n,r),(s,t-1,0)}^{x,D-x,x,D-x}. \end{aligned}$$

Thus we use Theorem 4.18 to determine each of the values, which leads to the statement. \square

Corollary. For any $(m, n, r) \in L'$, the matrix $\varphi_{m,n,r}(A)$ is irreducible tridiagonal, that is a tridiagonal matrix such that all of the subdiagonal entries and the superdiagonal entries are nonzero.

Theorem 4.21. For $(m, n, r) \in L'$, the representations $\varphi_{m,n,r}$ are irreducible as representations of \mathcal{T} .

Proof. Fix $(m, n, r) \in L'$ and set $d = d(m, n, r)$ and $\nu = \nu(m, n, r)$. Let $\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_d$ be the standard basis of \mathbb{C}^{d+1} . Take any nonzero vector $\mathbf{v} \in \mathbb{C}^{d+1}$ and write $\mathbf{v} = \alpha_0 \mathbf{e}_0 + \alpha_1 \mathbf{e}_1 + \dots + \alpha_d \mathbf{e}_d$. Since $\mathbf{v} \neq \mathbf{0}$, there exists a nonzero element α_j . By $\varphi_{m,n,r}(E_{j+\nu}^*)\mathbf{v} = \alpha_j \mathbf{e}_j$, we have $\mathbf{e}_j \in \varphi_{m,n,r}(\mathcal{T})\mathbf{v}$. By the corollary above, we have $\varphi_{m,n,r}(A)\mathbf{e}_j = b_{j-1}\mathbf{e}_{j-1} + a_j\mathbf{e}_j + c_{j+1}\mathbf{e}_{j+1} \in \varphi_{m,n,r}(\mathcal{T})\mathbf{v}$ for some scalars b_{j-1}, a_j, c_{j+1} where b_{j-1}, c_{j+1} are nonzero, and therefore $\mathbf{e}_{j\pm 1} \in \varphi_{m,n,r}(\mathcal{T})\mathbf{v}$. Finally, we obtain $\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_d \in \varphi_{m,n,r}(\mathcal{T})\mathbf{v}$ by iteration. \square

By the general theory of matrix $*$ -algebras, it is known that every irreducible representation of \mathcal{T} must arise as a subrepresentation of the representation on $L(X_D)$ (cf. Gijswijt [10]). Therefore, the $\varphi_{m,n,r}$ exhaust all the (isomorphism classes of) irreducible representations of \mathcal{T} . Thus we have determined all the irreducible representations up to this point.

For additional information, we view relations with Leonard pairs. For a vector space V , a pair of linear transformations $B : V \rightarrow V$ and $B^* : V \rightarrow V$ is called a Leonard pair if B has an irreducible tridiagonal matrix representation and B^* has a diagonal matrix representation with respect to a common basis of V , and dually, B^* has an irreducible tridiagonal matrix representation and B has a diagonal matrix representation with respect to another basis of V . The point is that Leonard pairs are classified up to isomorphism. In our case, we determined that there exists $A^* \in \mathcal{T}$, which we call a dual adjacency matrix, such that $(\varphi_{m,n,r}(A), \varphi_{m,n,r}(A^*))$ is a Leonard pair of Type I (by Terwilliger [26]) for every $(m, n, r) \in L'$ with the following parameters:

- (i) Suppose $D - b \leq m - n \leq a - D$.

$$\begin{aligned} d &= D - m - n - r, \quad h = \frac{q^{a+b-m-n-r+1}}{(q-1)^2}, \\ r_1 &= 0, \quad r_2 = q^{-b+2n+r-1}, \quad s = q^{-a-b+2m+2n+2r-2}, \quad s^* = 0, \\ \theta_0 &= \frac{1}{(q-1)^2} \left(q - 1 - q^{D+1} - 1^{-D+a+b+1} + q^{a+b-m-n-r+1} + q^{m+n+r} \right). \end{aligned}$$

- (ii) Suppose $m - n \leq \min\{a - D, D - b\}$.

$$\begin{aligned} d &= b - 2n - r, \quad h = \frac{q^{a+b-m-n-r+1}}{(q-1)^2}, \\ r_1 &= 0, \quad r_2 = q^{-D+m+n+r-1}, \quad s = q^{-a-b+2m+2n+2r-2}, \quad s^* = 0, \\ \theta_0 &= \frac{1}{(q-1)^2} \left(q - 1 - q^{D+1} - 1^{-D+a+b+1} + q^{a+b-m-n-r+1} + q^{m+n+r} \right). \end{aligned}$$

- (iii) Suppose $\max\{a - D, D - b\} \leq m - n$.

$$\begin{aligned} d &= a - 2m - r, \quad h = \frac{q^{a+b-m-n-r+1}}{(q-1)^2}, \\ r_1 &= 0, \quad r_2 = q^{D-a-b+m+n+r-1}, \quad s = q^{-a-b+2m+2n+2r-2}, \quad s^* = 0, \\ \theta_0 &= \frac{1}{(q-1)^2} \left(q - 1 - q^{D+1} - 1^{-D+a+b+1} + q^{a+b-m-n-r+1} + q^{m+n+r} \right). \end{aligned}$$

- (iv) Suppose $a - D \leq m - n \leq D - b$.

$$\begin{aligned} d &= -D + a + b - m - n - r, \quad h = \frac{q^{a+b-m-n-r+1}}{(q-1)^2}, \\ r_1 &= 0, \quad r_2 = q^{-a+2m+r-1}, \quad s = q^{-a-b+2m+2n+2r-2}, \quad s^* = 0, \\ \theta_0 &= \frac{1}{(q-1)^2} \left(q - 1 - q^{D+1} - 1^{-D+a+b+1} + q^{a+b-m-n-r+1} + q^{m+n+r} \right). \end{aligned}$$

Note that the parameters except d and r_2 are the same for all cases. In particular, if we assume $a = D$, this result agrees with the result listed in [28].

4.4 The Erdős-Ko-Rado Theorem for Singular Linear Spaces

As a second application of the structure of the coherent configuration on the finite projective geometry, we introduce the Erdős-Ko-Rado theorem for singular linear spaces. Singular linear spaces are the sets $X_{i,j}$ which are defined in (4.7). We fix one singular linear space $X_{i,j}$. A subset of $X_{i,j}$ is called t -intersecting if any two elements of the subset have at least t -dimensional intersection, i.e., for any ξ, η from the subset,

$$\dim(\xi \cap \eta) \geq t.$$

We want to evaluate the maximal size of such a subset, which has a t -intersecting property. The Erdős-Ko-Rado theorem for $X_{i,j}$ is the answer of the question and partially proved by Ou, Lv and Wang in [19]. Remark that we do not want to consider the obvious cases such as $X_{a,0}$ and $X_{a,b}$, both of which consist of only one element. In addition we have the obvious restriction for t :

$$t \leq \dim(\xi \cap \eta) \leq \dim \xi = i + j \quad (\xi, \eta \in X_{i,j}). \quad (4.39)$$

If equality holds, the subspaces ξ and η are the same. In order to avoid such obvious cases, we assume $1 \leq t < i + j$ and $(i, j) \neq (a, 0), (a, b)$.

Theorem 4.22 (Ou-Lv-Wang [19]). *Let \mathcal{F} be a t -intersecting family in $X_{i,j}$.*

(i) *Assume that $j < b$, $2i + j + 1 \leq a$, $t < i$ and $(a, q) \neq (2i + j + 1, 2)$. Then*

$$|\mathcal{F}| \leq q^{(a-i)j} \begin{bmatrix} b \\ j \end{bmatrix}_q \begin{bmatrix} a-t \\ i-t \end{bmatrix}_q,$$

and equality holds if and only if \mathcal{F} consists of all elements containing a fixed subspace $\zeta \in X_{t,0}$.

(ii) *Assume that $j < b$, $2i + 2j + 1 \leq a + b$, $i \leq t$ and $(a+b, q) \neq (2i + 2j + 1, 2)$. Then*

$$|\mathcal{F}| \leq q^{(a-i)(i+j-t)} \begin{bmatrix} b+i-t \\ i+j-t \end{bmatrix}_q,$$

and equality holds if and only if \mathcal{F} consists of all elements containing a fixed subspace $\zeta \in X_{i,t-i}$.

(iii) *Assume that $b = j$, $2i + 2j + 1 \leq a + b$, $t < j$ and $(a+b, q) \neq (2i + 2j + 1, 2)$. Then*

$$|\mathcal{F}| \leq q^{(a-i)(j-t)} \begin{bmatrix} a \\ i \end{bmatrix}_q,$$

and equality holds if and only if \mathcal{F} consists of all elements containing a fixed subspace $\zeta \in X_{0,t}$.

(iv) *Assume that $b = j$, $2i + 2j + 1 \leq a + b$, $j \leq t$ and $(a+b, q) \neq (2i + 2j + 1, 2)$. Then*

$$|\mathcal{F}| \leq \begin{bmatrix} a+j-t \\ i+j-t \end{bmatrix}_q,$$

and equality holds if and only if \mathcal{F} consists of all elements containing a fixed subspace $\zeta \in X_{t-j,j}$.

We called the Erdős-Ko-Rado theorem “partially” proved because Theorem 4.22 covers only special parameters. Here, we use the association scheme based on $X_{i,j}$ which is discussed in this chapter, and Delsarte’s linear programming method, which is discussed in Chapter 3. Then we wish to obtain the Erdős-Ko-Rado theorem for $X_{i,j}$ without any parameter restrictions. Remark that our approach is totally different from their combinatorial way.

Theorem 4.23. *Let $(X_{i,j}, \{R_{v,w,z}\}_{(v,w,z) \in K})$ be the association scheme based on $X_{i,j}$. Let $\mathbf{a} = (a_{v,w,z})_{(v,w,z) \in K}$ be the inner distribution of a family \mathcal{F} in $X_{i,j}$. Then the following are equivalent.*

(i) \mathcal{F} is t -intersecting.

(ii) $a_{v,w,z} = 0$ for any $(v, w, z) \in K$ with $0 \leq v + w - z < t$.

Proof. This follows from Theorem 4.4. \square

Theorem 4.24. Let $Q = \left(Q_{(v,w,z),(m,n,r)}^{i,j,i,j} \right)$ be the second eigenmatrix of the association scheme based on $X_{i,j}$. Define

$$a^* = \max \sum_{(v,w,z) \in K} a_{v,w,z}, \quad \text{subject to} \quad \begin{cases} a_{i,j,0} = 1, \\ a_{v,w,z} \geq 0 & \text{for } (v, w, z) \in K, \\ a_{v,w,z} = 0 & \text{for } (v, w, z) \in K \text{ with } 0 \leq v + w - z < t, \\ (\mathbf{a}Q)_{m,n,r} \geq 0 & \text{for } (m, n, r) \in L. \end{cases}$$

Then a^* is an upper bound for the size of a t -intersecting family in $X_{i,j}$.

Yet, we have not guaranteed that the upper bound in Theorem 4.24 coincides with the one in Theorem 4.22 in general. However we have checked that these bounds are the same for the small parameters such that $1 \leq i \leq a \leq 10$, $1 \leq j \leq b \leq 10$ and $q = 2, 3$ by mathematica computation¹. We have the exact bounds for some small parameters in Appendix C.2.

¹By Mathematica 9.0 for Mac OS X x86 (64-bit) in Mac OS X 10.9 (1.6 GHz Intel Core i5).

Appendix A

Some Formulas for Binomial Coefficients

This chapter will provide some basic properties and formulas of binomial coefficients which we need in this paper. Contents in this chapter are based on the textbook [20]. See the book for more information.

Binomial coefficients are first defined for integers n and m (not necessarily positive) by

$$\binom{n}{m} = \begin{cases} \frac{n!}{m!(n-m)!} & \text{if } 0 \leq m \leq n, \\ (-1)^m \frac{(-n+m-1)!}{m!(-n-1)!} & \text{if } n < 0 \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

By definition, we have for any integers n and m ,

$$\binom{n+m-1}{m} = (-1)^m \binom{-n}{m}. \quad (\text{A.1})$$

The following relations for binomial coefficients are well known and indispensable: For any integers n, m, p , we have

$$\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1}, \quad (\text{A.2})$$

$$\binom{n}{m} = \binom{n}{n-m} \quad (\text{if } n \geq 0), \quad (\text{A.3})$$

$$\binom{n}{m} \binom{m}{p} = \binom{n}{p} \binom{n-p}{m-p}. \quad (\text{A.4})$$

Theorem A.1 (Vandermonde Convolution Formula). *For any integers n, m and any nonnegative integer p , we have*

$$\binom{n}{m} = \sum_{k=0}^m \binom{n-p}{m-k} \binom{p}{k}.$$

Proof. This will be obtained by applying (A.2) repeatedly. \square

The next formulas are used in Chapter 1.

Theorem A.2. (i) *Assume n, t are nonnegative integers. Then we have*

$$\binom{n}{m-t} = \sum_{j=0}^t (-1)^{t-j} \binom{t}{j} \binom{n+j}{m}.$$

(ii) For any integers i, k, r, u such that $u \geq i$, we have

$$\binom{k-u}{r-i} \binom{u}{i} = \sum_{t=i}^u (-1)^{t-i} \binom{t}{i} \binom{k-t}{r-t} \binom{u}{t}$$

Proof. At first, we will prepare the formula:

$$\begin{aligned} \binom{n-p}{m} &= (-1)^m \binom{-n+p+m-1}{m} && \text{by (A.1)} \\ &= (-1)^m \sum_{k=0}^m \binom{-n+m-1}{m-k} \binom{p}{k} && \text{by Theorem A.1} \\ &= (-1)^m \sum_{k=0}^m (-1)^{m-k} \binom{n-k}{m-k} \binom{p}{k} && \text{by (A.1)} \\ &= \sum_{k=0}^m (-1)^k \binom{n-k}{m-k} \binom{p}{k}. && (\star) \end{aligned}$$

- (i) The value $\binom{t}{j}$ will be zero if $j < 0$ and the value $\binom{n+j}{m}$ will be zero if $-n \leq j < m-n$. Then we change the range of the summation;

$$\begin{aligned} &\sum_{j=0}^t (-1)^{t-j} \binom{t}{j} \binom{n+j}{m} \\ &= \sum_{j=m-n}^t (-1)^{t-j} \binom{t}{j} \binom{n+j}{m} \\ &= \sum_{j=m-n}^t (-1)^{t-j} \binom{t}{t-j} \binom{n+j}{m} && \text{by (A.3)} \\ &= \sum_{k=0}^{t-(m-n)} (-1)^k \binom{t}{k} \binom{n+t-k}{m} && k = t-j \\ &= \sum_{k=0}^{n+t-m} (-1)^k \binom{t}{k} \binom{n+t-k}{n+t-m-k} && \text{by (A.3)} \\ &= \binom{(n+t)-t}{n+t-m} && \text{by } (\star) \\ &= \binom{n}{n+t-m} \\ &= \binom{n}{m-t} && \text{by (A.3).} \end{aligned}$$

(ii)

$$\begin{aligned} &\sum_{t=i}^u (-1)^{t-i} \binom{t}{i} \binom{k-t}{r-t} \binom{u}{t} \\ &= \sum_{t=i}^u (-1)^{t-i} \binom{k-t}{r-t} \binom{u}{t} \binom{t}{i} \\ &= \sum_{t=i}^u (-1)^{t-i} \binom{k-t}{r-t} \binom{u}{i} \binom{u-i}{t-i} && \text{by (A.4)} \\ &= \sum_{s=0}^{u-i} (-1)^s \binom{k-i-s}{r-i-s} \binom{u}{i} \binom{u-i}{s} && s = t-i \\ &= \binom{k-u}{r-i} \binom{u}{i} && \text{by } (\star). \end{aligned}$$

□

Appendix B

Mathematica Codes

B.1 The Bounds in Theorem 3.9

```
(* - - - - - Define the second eigenmatrix Q - - - - - *)
P[i_,j_]:=Sum[(-1)^(j-k)*Binomial[d-k,j-k]
               *Binomial[d-i,k]*Binomial[m-d+k-i,k],{k,0,j}];

Mu[i_]:=Binomial[m,i]-Binomial[m,i-1];
Nu[j_]:=Binomial[d,j]*Binomial[m-d,j];
Q[j_,i_]:=Mu[i]/Nu[j]*P[i,j];

(* - - - - - Evaluate the bound - - - - - *)
Mybound[t_,v_,k_]:=(
m:=v;
d:=k;
e:=Floor[t/2];
fis:=Binomial[v,e];
Ceiling[Max[Table[
  fis+Q[0,mm]/fis*
  Min[Table[
    (Sum[Q[i,l],{l,0,e}])^2/Abs[Q[i,mm]]
    ,{i, Select[Table[i,{i,1,k}],Q[#,mm]<0&]}]
  ,{mm,1,t}]]]
)
```

B.2 The Linear Programming Bounds in Theorem 4.24

```
(* - - - - - Define some polynomials - - - - - *)
QEberlein[m_,a_,b_,c_,x_,q_]:=q^(m*(m-1)/2)*QPochhammer[q^a,q^(-1),m]*QPochhammer[q^c,q^(-1),m]*
Sum[
  QPochhammer[q^m,q^(-1),j]*QPochhammer[q^(a+b-m+1),q^(-1),j]*
  QPochhammer[q^x,q^(-1),j]*q^(-j)/
  (QPochhammer[q^a,q^(-1),j]*QPochhammer[q^c,q^(-1),j]
   *QPochhammer[q^(-1),q^(-1),j]),
  {j,0,Min[m,a+b-m+1,a,c]}];

QKrawtchouk[m_,x_,a_,b_,q_]:=QPochhammer[q^a,q^(-1),m]*QPochhammer[q^b,q^(-1),m]/
QPochhammer[q^(-1),q^(-1),m]*q^(-m)*
Sum[
  QPochhammer[q^(-m),q,j]*QPochhammer[q^(-x),q,j]*q^j/
  (QPochhammer[q^(-a),q,j]*QPochhammer[q^(-b),q,j]
   *QPochhammer[q,q,j])]
```

```

,{j,0,Min[m,x,a,b]}];

(* ----- Define the bound by EKR theorem from Ou et. al.----- *)
EKR:=
If[q==2,
Which[
j<b && 2*i+j+1 < a && t<i,
q^((a-i)*j)*QBinomial[b,j,q]*QBinomial[a-t,i-t,q],
j<b && 2*i+2*j+1 < a+b && i <= t,
q^((a-i)*(i+j-t))*QBinomial[b+i-t,i+j-t,q],
b==j && 2*i+2*j+1<a+b && t<j,
q^((a-i)*(j-t))*QBinomial[a,i,q],
b==j && 2*i+2*j+1<a+b && j<=t,
QBinomial[a+j-t,i+j-t,q],
True,
0
]
,Which[
j<b && 2*i+j+1 <=a && t<i,
q^((a-i)*j)*QBinomial[b,j,q]*QBinomial[a-t,i-t,q],
j<b && 2*i+2*j+1 <= a+b && i <= t,
q^((a-i)*(i+j-t))*QBinomial[b+i-t,i+j-t,q],
b==j && 2*i+2*j+1<=a+b && t<j,
q^((a-i)*(j-t))*QBinomial[a,i,q],
b==j && 2*i+2*j+1<=a+b && j<=t,
QBinomial[a+j-t,i+j-t,q],
True,
0
]
];
(* ----- Define the the second eigenmatrix Q ----- *)
dimV[m_,n_,r_]:= 
QPochhammer[q^a,q^(-1),r]*QPochhammer[q^b,q^(-1),r]/
QPochhammer[q^(-1),q^(-1),r]*q^(-r)*
QBinomial[a-r,m,q]*QBinomial[b-r,n,q]*
(q^m-q^(a-r-m+1))/(1-q^(a-r-m+1))*(q^n-q^(b-r-n+1))/(1-q^(b-r-n+1));

Q[m_,n_,r_,v_,w_,z_]:= 
QEberlein[m,i,a-r-i,i,v,q]*QEberlein[n,j-r,b-j,j-r,w-r,q]*
QKrawtchouk[r,z,w,a-2*i+v,q]/
(QEberlein[m,i,a-r-i,i,i,q]*QEberlein[n,j-r,b-j,j-r,j-r,q]*
QKrawtchouk[r,0,j,a-i,q])*dimV[m,n,r];

QMatrix:=
Flatten[Table[
Flatten[Table[
Q[m,n,r,v,w,z]
,{r,0,Min[a-i,j]}, {m,0,Min[i,a-r-i]}, {n,0,Min[j-r,b-j]}]
,{v,Max[0,-a+2*i],i}, {w,Max[0,-b+2*j],j}, {z,0,Min[w,a-2*i+v]}],2];

(* ----- Construct a linear programming problem ----- *)
CharacteristicVector:=
Flatten[Table[
Which[v==i && w==j && z==0, 0, v+w-z<t, 0, True, 1]
,{v,Max[0,-a+2*i],i}, {w,Max[0,-b+2*j],j}, {z,0,Min[w,a-2*i+v]}]];

l:=Total[CharacteristicVector];

```

```
c:=Table[-1,{x,1,1}];  
mm:=Transpose[Pick[QMatrix, CharacteristicVector, 1]];  
bb:=-Flatten[Table[Q[m,n,r,i,j,0]  
,{r,0,Min[a-i,j]}, {m,0,Min[i,a-r-i]}, {n,0,Min[j-r,b-j]}]];  
MyEKR := Check[Total[LinearProgramming[c,mm,bb]]+1,0]
```

Appendix C

Tables

C.1 Table of Lower Bounds by Theorem 3.9

Table C.1: Lower Bounds on # of blocks of t - (v, k, λ) designs ($2 \leq t \leq 6, v \leq 70$)

t	v	k	λ	# of blocks	Fisher	Thm 3.9	Delsarte
2	16	6	1	8	16	16	16
2	21	6	1	14	21	26	28
2	25	10	3	20	25	28	28
2	34	12	2	17	34	34	34
2	36	15	4	24	36	36	36
2	45	12	2	30	45	45	45
2	46	10	1	23	46	46	46
2	49	21	5	28	49	52	52
2	52	18	3	26	52	52	52
2	55	10	1	33	55	60	66
2	55	22	7	45	55	59	60
2	57	21	5	38	57	62	62
2	64	28	9	48	64	64	64
2	66	26	5	33	66	66	66
2	69	18	3	46	69	76	77
2	70	24	4	35	70	70	70
3	22	7	1	44	22	64	88
3	56	11	1	168	56	237	336
4	12	6	2	66	66	74	110
4	17	7	2	136	136	154	204
4	23	8	2	253	253	293	355
4	23	11	6	161	253	260	337
4	38	10	2	703	703	766	904
4	42	16	14	861	861	902	1062
4	47	11	2	1081	1081	1153	1352
4	47	23	35	705	1081	1109	1217
4	57	12	2	1596	1596	1679	1951
4	57	21	19	1254	1596	1629	1840
4	58	22	19	1102	1653	1663	1764
6	19	9	2	646	969	975	3553
6	39	18	52	9139	9139	9190	11001
6	49	24	133	13818	18424	19098	23602
6	53	17	13	24115	23426	24631	31832
6	59	29	260	24662	32509	32566	36905

C.2 Table of Upper Bounds by Theorem 4.24

Here we list the upper bounds for a t -intersecting family on $X_{i,j}$ from Theorem 4.24 for all parameters satisfying:

$$0 \leq i \leq a \leq 5, \quad 0 \leq j \leq b \leq 5, \quad 1 \leq t \leq \min\{i+j-1, 5\}, \quad q = 2, 3.$$

We have computed the bounds for much more parameters, but we list only some of them due to the limit of the available pages. In order to compare bounds from L. Ou, B. Lv and K. Wang (Theorem 4.22), we list them together. Blanks mean that the parameters are not suitable for their theorem.

a	b	i	j	t	q	Thm 4.24	Thm 4.22
1	2	0	2	1	2	4	
1	2	0	2	1	3	9	
1	2	1	1	1	2	3	
1	2	1	1	1	3	4	
1	3	0	2	1	2	7	
1	3	0	2	1	3	13	
1	3	0	3	1	2	8	
1	3	0	3	1	3	27	
1	3	0	3	2	2	8	
1	3	0	3	2	3	27	
1	3	1	1	1	2	7	
1	3	1	1	1	3	13	
1	3	1	2	1	2	7	
1	3	1	2	1	3	13	
1	3	1	2	2	2	7	
1	3	1	2	2	3	13	
1	4	0	2	1	2	14	
1	4	0	2	1	3	39	39
1	4	0	3	1	2	120	
1	4	0	3	1	3	1080	
1	4	0	3	2	2	15	
1	4	0	3	2	3	40	
1	4	0	4	1	2	16	
1	4	0	4	1	3	81	
1	4	0	4	2	2	16	
1	4	0	4	2	3	81	
1	4	0	4	3	2	16	
1	4	0	4	3	3	81	
1	4	1	1	1	2	15	
1	4	1	1	1	3	40	40
1	4	1	2	1	2	35	
1	4	1	2	1	3	130	
1	4	1	2	2	2	7	
1	4	1	2	2	3	13	
1	4	1	3	1	2	15	
1	4	1	3	1	3	40	
1	4	1	3	2	2	15	
1	4	1	3	2	3	40	
1	4	1	3	3	2	15	
1	4	1	3	3	3	40	
1	5	0	2	1	2	30	30
1	5	0	2	1	3	120	120
1	5	0	3	1	2	155	
1	5	0	3	1	3	1210	
1	5	0	3	2	2	15	
1	5	0	3	2	3	40	

a	b	i	j	t	q	Thm 4.24	Thm 4.22
1	5	0	4	1	2	496	
1	5	0	4	1	3	9801	
1	5	0	4	2	2	496	
1	5	0	4	2	3	9801	
1	5	0	4	3	2	31	
1	5	0	4	3	3	121	
1	5	0	5	1	2	32	
1	5	0	5	1	3	243	
1	5	0	5	2	2	32	
1	5	0	5	2	3	243	
1	5	0	5	3	2	32	
1	5	0	5	3	3	243	
1	5	0	5	4	2	32	
1	5	0	5	4	3	243	
1	5	1	1	1	2	31	31
1	5	1	1	1	3	121	121
1	5	1	2	1	2	155	
1	5	1	2	1	3	1210	
1	5	1	2	2	2	15	
1	5	1	2	2	3	40	
1	5	1	3	1	2	155	
1	5	1	3	1	3	1210	
1	5	1	3	2	2	155	
1	5	1	3	2	3	1210	
1	5	1	3	3	2	15	
1	5	1	3	3	3	40	
1	5	1	4	1	2	31	
1	5	1	4	1	3	121	
1	5	1	4	2	2	31	
1	5	1	4	2	3	121	
1	5	1	4	3	2	31	
1	5	1	4	3	3	121	
1	5	1	4	4	2	31	
1	5	1	4	4	3	121	
2	1	1	1	1	2	6	
2	1	1	1	1	3	12	
2	2	0	2	1	2	4	
2	2	0	2	1	3	9	
2	2	1	1	1	2	6	
2	2	1	1	1	3	12	
2	2	1	2	1	2	12	
2	2	1	2	1	3	36	
2	2	1	2	2	2	12	
2	2	1	2	2	3	36	
2	2	2	1	1	2	3	
2	2	2	1	1	3	4	
2	2	2	1	2	2	3	
2	2	2	1	2	3	4	
2	3	0	2	1	2	12	
2	3	0	2	1	3	36	36
2	3	0	3	1	2	64	
2	3	0	3	1	3	729	
2	3	0	3	2	2	8	
2	3	0	3	2	3	27	
2	3	1	1	1	2	14	
2	3	1	1	1	3	39	39
2	3	1	2	1	2	84	

a	b	i	j	t	q	Thm 4.24	Thm 4.22
2	3	1	2	1	3	468	
2	3	1	2	2	2	12	
2	3	1	2	2	3	36	
2	3	1	3	1	2	24	
2	3	1	3	1	3	108	
2	3	1	3	2	2	24	
2	3	1	3	2	3	108	
2	3	1	3	3	2	24	
2	3	1	3	3	3	108	
2	3	2	1	1	2	7	
2	3	2	1	1	3	13	
2	3	2	1	2	2	7	
2	3	2	1	2	3	13	
2	3	2	2	1	2	7	
2	3	2	2	1	3	13	
2	3	2	2	2	2	7	
2	3	2	2	2	3	13	
2	3	2	2	3	2	7	
2	3	2	2	3	3	13	
2	4	0	2	1	2	28	28
2	4	0	2	1	3	117	117
2	4	0	3	1	2	120	
2	4	0	3	1	3	1080	
2	4	0	3	2	2	15	
2	4	0	3	2	3	40	
2	4	0	4	1	2	256	
2	4	0	4	1	3	6561	
2	4	0	4	2	2	256	
2	4	0	4	2	3	6561	
2	4	0	4	3	2	16	
2	4	0	4	3	3	81	
2	4	1	1	1	2	30	30
2	4	1	1	1	3	120	120
2	4	1	2	1	2	140	
2	4	1	2	1	3	1170	
2	4	1	2	2	2	14	
2	4	1	2	2	3	39	
2	4	1	3	1	2	360	
2	4	1	3	1	3	4320	
2	4	1	3	2	2	360	
2	4	1	3	2	3	4320	
2	4	1	3	3	2	24	
2	4	1	3	3	3	108	
2	4	1	4	1	2	48	
2	4	1	4	1	3	324	
2	4	1	4	2	2	48	
2	4	1	4	2	3	324	
2	4	1	4	3	2	48	
2	4	1	4	3	3	324	
2	4	1	4	4	2	48	
2	4	1	4	4	3	324	
2	4	2	1	1	2	15	
2	4	2	1	1	3	40	
2	4	2	1	2	2	15	
2	4	2	1	2	3	40	
2	4	2	2	1	2	35	
2	4	2	2	1	3	130	

a	b	i	j	t	q	Thm 4.24	Thm 4.22
2	4	2	2	2	2	35	
2	4	2	2	2	3	130	
2	4	2	2	3	2	7	
2	4	2	2	3	3	13	
2	4	2	3	1	2	15	
2	4	2	3	1	3	40	
2	4	2	3	2	2	15	
2	4	2	3	2	3	40	
2	4	2	3	3	2	15	
2	4	2	3	3	3	40	
2	4	2	3	4	2	15	
2	4	2	3	4	3	40	
2	5	0	2	1	2	60	60
2	5	0	2	1	3	360	360
2	5	0	3	1	2	560	
2	5	0	3	1	3	10530	10530
2	5	0	3	2	2	28	
2	5	0	3	2	3	117	117
2	5	0	4	1	2	7936	
2	5	0	4	1	3	793881	
2	5	0	4	2	2	496	
2	5	0	4	2	3	9801	
2	5	0	4	3	2	31	
2	5	0	4	3	3	121	
2	5	0	5	1	2	1024	
2	5	0	5	1	3	59049	
2	5	0	5	2	2	1024	
2	5	0	5	2	3	59049	
2	5	0	5	3	2	1024	
2	5	0	5	3	3	59049	
2	5	0	5	4	2	32	
2	5	0	5	4	3	243	
2	5	1	1	1	2	62	62
2	5	1	1	1	3	363	363
2	5	1	2	1	2	620	
2	5	1	2	1	3	10890	10890
2	5	1	2	2	2	30	
2	5	1	2	2	3	120	120
2	5	1	3	1	2	3720	
2	5	1	3	1	3	130680	
2	5	1	3	2	2	360	
2	5	1	3	2	3	4320	
2	5	1	3	3	2	24	
2	5	1	3	3	3	108	
2	5	1	4	1	2	1488	
2	5	1	4	1	3	39204	
2	5	1	4	2	2	1488	
2	5	1	4	2	3	39204	
2	5	1	4	3	2	1488	
2	5	1	4	3	3	39204	
2	5	1	4	4	2	48	
2	5	1	4	4	3	324	
2	5	1	5	1	2	96	
2	5	1	5	1	3	972	
2	5	1	5	2	2	96	
2	5	1	5	2	3	972	
2	5	1	5	3	2	96	

a	b	i	j	t	q	Thm 4.24	Thm 4.22
2	5	1	5	3	3	972	
2	5	1	5	4	2	96	
2	5	1	5	4	3	972	
2	5	1	5	5	2	96	
2	5	1	5	5	3	972	
2	5	2	1	1	2	31	
2	5	2	1	1	3	121	
2	5	2	1	2	2	31	
2	5	2	1	2	3	121	121
2	5	2	2	1	2	155	
2	5	2	2	1	3	1210	
2	5	2	2	2	2	155	
2	5	2	2	2	3	1210	
2	5	2	2	3	2	15	
2	5	2	2	3	3	40	
2	5	2	3	1	2	155	
2	5	2	3	1	3	1210	
2	5	2	3	2	2	155	
2	5	2	3	2	3	1210	
2	5	2	3	3	2	155	
2	5	2	3	3	3	1210	
2	5	2	3	4	2	15	
2	5	2	3	4	3	40	
2	5	2	4	1	2	31	
2	5	2	4	1	3	121	
2	5	2	4	2	2	31	
2	5	2	4	2	3	121	
2	5	2	4	3	2	31	
2	5	2	4	3	3	121	
2	5	2	4	4	2	31	
2	5	2	4	4	3	121	
2	5	2	4	5	2	31	
2	5	2	4	5	3	121	
3	1	1	1	1	2	7	
3	1	1	1	1	3	13	
3	1	2	0	1	2	7	
3	1	2	0	1	3	13	
3	1	2	1	1	2	14	
3	1	2	1	1	3	39	
3	1	2	1	2	2	14	
3	1	2	1	2	3	39	
3	2	0	2	1	2	8	
3	2	0	2	1	3	27	27
3	2	1	1	1	2	12	
3	2	1	1	1	3	36	36
3	2	1	2	1	2	112	
3	2	1	2	1	3	1053	
3	2	1	2	2	2	12	
3	2	1	2	2	3	36	
3	2	2	0	1	2	7	
3	2	2	0	1	3	13	
3	2	2	1	1	2	42	
3	2	2	1	1	3	156	
3	2	2	1	2	2	14	
3	2	2	1	2	3	39	
3	2	2	2	1	2	28	
3	2	2	2	1	3	117	

a	b	i	j	t	q	Thm 4.24	Thm 4.22
3	2	2	2	2	2	28	
3	2	2	2	2	3	117	
3	2	2	2	3	2	28	
3	2	2	2	3	3	117	
3	2	3	1	1	2	3	
3	2	3	1	1	3	4	
3	2	3	1	2	2	3	
3	2	3	1	2	3	4	
3	2	3	1	3	2	3	
3	2	3	1	3	3	4	
3	3	0	2	1	2	24	24
3	3	0	2	1	3	108	108
3	3	0	3	1	2	64	
3	3	0	3	1	3	729	
3	3	0	3	2	2	8	
3	3	0	3	2	3	27	
3	3	1	1	1	2	28	28
3	3	1	1	1	3	117	117
3	3	1	2	1	2	112	
3	3	1	2	1	3	1053	
3	3	1	2	2	2	12	
3	3	1	2	2	3	36	
3	3	1	3	1	2	448	
3	3	1	3	1	3	9477	
3	3	1	3	2	2	448	
3	3	1	3	2	3	9477	
3	3	1	3	3	2	24	
3	3	1	3	3	3	108	
3	3	2	0	1	2	7	
3	3	2	0	1	3	13	
3	3	2	1	1	2	98	
3	3	2	1	1	3	507	
3	3	2	1	2	2	14	
3	3	2	1	2	3	39	
3	3	2	2	1	2	196	
3	3	2	2	1	3	1521	
3	3	2	2	2	2	196	
3	3	2	2	2	3	1521	
3	3	2	2	3	2	28	
3	3	2	2	3	3	117	
3	3	2	3	1	2	56	
3	3	2	3	1	3	351	
3	3	2	3	2	2	56	
3	3	2	3	2	3	351	
3	3	2	3	3	2	56	
3	3	2	3	3	3	351	
3	3	2	3	4	2	56	
3	3	2	3	4	3	351	
3	3	3	1	1	2	7	
3	3	3	1	1	3	13	
3	3	3	1	2	2	7	
3	3	3	1	2	3	13	
3	3	3	1	3	2	7	
3	3	3	1	3	3	13	
3	3	3	2	1	2	7	
3	3	3	2	1	3	13	
3	3	3	2	2	2	7	

a	b	i	j	t	q	Thm 4.24	Thm 4.22
3	3	3	2	2	3	13	
3	3	3	2	3	2	7	
3	3	3	2	3	3	13	
3	3	3	2	4	2	7	
3	3	3	2	4	3	13	
3	4	0	2	1	2	56	56
3	4	0	2	1	3	351	351
3	4	0	3	1	2	448	
3	4	0	3	1	3	9477	9477
3	4	0	3	2	2	24	
3	4	0	3	2	3	108	108
3	4	0	4	1	2	4096	
3	4	0	4	1	3	531441	
3	4	0	4	2	2	256	
3	4	0	4	2	3	6561	
3	4	0	4	3	2	16	
3	4	0	4	3	3	81	
3	4	1	1	1	2	60	60
3	4	1	1	1	3	360	360
3	4	1	2	1	2	560	
3	4	1	2	1	3	10530	10530
3	4	1	2	2	2	28	
3	4	1	2	2	3	117	117
3	4	1	3	1	2	6720	
3	4	1	3	1	3	379080	
3	4	1	3	2	2	448	
3	4	1	3	2	3	9477	
3	4	1	3	3	2	24	
3	4	1	3	3	3	108	
3	4	1	4	1	2	1792	
3	4	1	4	1	3	85293	
3	4	1	4	2	2	1792	
3	4	1	4	2	3	85293	
3	4	1	4	3	2	1792	
3	4	1	4	3	3	85293	
3	4	1	4	4	2	48	
3	4	1	4	4	3	324	
3	4	2	0	1	2	7	
3	4	2	0	1	3	13	
3	4	2	1	1	2	210	
3	4	2	1	1	3	1560	
3	4	2	1	2	2	30	
3	4	2	1	2	3	120	120
3	4	2	2	1	2	980	
3	4	2	2	1	3	15210	
3	4	2	2	2	2	196	
3	4	2	2	2	3	1521	
3	4	2	2	3	2	28	
3	4	2	2	3	3	117	
3	4	2	3	1	2	840	
3	4	2	3	1	3	14040	
3	4	2	3	2	2	840	
3	4	2	3	2	3	14040	
3	4	2	3	3	2	840	
3	4	2	3	3	3	14040	
3	4	2	3	4	2	56	
3	4	2	3	4	3	351	

a	b	i	j	t	q	Thm 4.24	Thm 4.22
3	4	2	4	1	2	112	
3	4	2	4	1	3	1053	
3	4	2	4	2	2	112	
3	4	2	4	2	3	1053	
3	4	2	4	3	2	112	
3	4	2	4	3	3	1053	
3	4	2	4	4	2	112	
3	4	2	4	4	3	1053	
3	4	2	4	5	2	112	
3	4	2	4	5	3	1053	
3	4	3	1	1	2	15	
3	4	3	1	1	3	40	
3	4	3	1	2	2	15	
3	4	3	1	2	3	40	
3	4	3	1	3	2	15	
3	4	3	1	3	3	40	
3	4	3	2	1	2	35	
3	4	3	2	1	3	130	
3	4	3	2	2	2	35	
3	4	3	2	2	3	130	
3	4	3	2	3	2	35	
3	4	3	2	3	3	130	
3	4	3	2	4	2	7	
3	4	3	2	4	3	13	
3	4	3	3	1	2	15	
3	4	3	3	1	3	40	
3	4	3	3	2	2	15	
3	4	3	3	2	3	40	
3	4	3	3	3	2	15	
3	4	3	3	3	3	40	
3	4	3	3	4	2	15	
3	4	3	3	4	3	40	
3	4	3	3	5	2	15	
3	4	3	3	5	3	40	
3	5	0	2	1	2	120	120
3	5	0	2	1	3	1080	1080
3	5	0	3	1	2	2240	2240
3	5	0	3	1	3	94770	94770
3	5	0	3	2	2	56	56
3	5	0	3	2	3	351	351
3	5	0	4	1	2	7936	
3	5	0	4	1	3	793881	
3	5	0	4	2	2	496	
3	5	0	4	2	3	9801	
3	5	0	4	3	2	31	
3	5	0	4	3	3	121	
3	5	0	5	1	2	32768	
3	5	0	5	1	3	14348907	
3	5	0	5	2	2	32768	
3	5	0	5	2	3	14348907	
3	5	0	5	3	2	1024	
3	5	0	5	3	3	59049	
3	5	0	5	4	2	32	
3	5	0	5	4	3	243	
3	5	1	1	1	2	124	124
3	5	1	1	1	3	1089	1089
3	5	1	2	1	2	2480	2480

a	b	i	j	t	q	Thm 4.24	Thm 4.22
3	5	1	2	1	3	98010	98010
3	5	1	2	2	2	60	60
3	5	1	2	2	3	360	360
3	5	1	3	1	2	9920	
3	5	1	3	1	3	882090	
3	5	1	3	2	2	560	
3	5	1	3	2	3	10530	
3	5	1	3	3	2	28	
3	5	1	3	3	3	117	
3	5	1	4	1	2	55552	
3	5	1	4	1	3	10320453	
3	5	1	4	2	2	55552	
3	5	1	4	2	3	10320453	
3	5	1	4	3	2	1792	
3	5	1	4	3	3	85293	
3	5	1	4	4	2	48	
3	5	1	4	4	3	324	
3	5	1	5	1	2	7168	
3	5	1	5	1	3	767637	
3	5	1	5	2	2	7168	
3	5	1	5	2	3	767637	
3	5	1	5	3	2	7168	
3	5	1	5	3	3	767637	
3	5	1	5	4	2	7168	
3	5	1	5	4	3	767637	
3	5	1	5	5	2	96	
3	5	1	5	5	3	972	
3	5	2	0	1	2	7	
3	5	2	0	1	3	13	
3	5	2	1	1	2	434	
3	5	2	1	1	3	4719	
3	5	2	1	2	2	62	
3	5	2	1	2	3	363	363
3	5	2	2	1	2	4340	
3	5	2	2	1	3	141570	
3	5	2	2	2	2	620	
3	5	2	2	2	3	10890	
3	5	2	2	3	2	30	
3	5	2	2	3	3	120	
3	5	2	3	1	2	8680	
3	5	2	3	1	3	424710	
3	5	2	3	2	2	8680	
3	5	2	3	2	3	424710	
3	5	2	3	3	2	840	
3	5	2	3	3	3	14040	
3	5	2	3	4	2	56	
3	5	2	3	4	3	351	
3	5	2	4	1	2	3472	
3	5	2	4	1	3	127413	
3	5	2	4	2	2	3472	
3	5	2	4	2	3	127413	
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3	5	2	4	3	3	127413	
3	5	2	4	4	2	3472	
3	5	2	4	4	3	127413	
3	5	2	4	5	2	112	
3	5	2	4	5	3	1053	

a	b	i	j	t	q	Thm 4.24	Thm 4.22
3	5	2	5	1	2	224	
3	5	2	5	1	3	3159	
3	5	2	5	2	2	224	
3	5	2	5	2	3	3159	
3	5	2	5	3	2	224	
3	5	2	5	3	3	3159	
3	5	2	5	4	2	224	
3	5	2	5	4	3	3159	
3	5	2	5	5	2	224	
3	5	2	5	5	3	3159	
3	5	3	1	1	2	31	
3	5	3	1	1	3	121	
3	5	3	1	2	2	31	
3	5	3	1	2	3	121	
3	5	3	1	3	2	31	
3	5	3	1	3	3	121	
3	5	3	2	1	2	155	
3	5	3	2	1	3	1210	
3	5	3	2	2	2	155	
3	5	3	2	2	3	1210	
3	5	3	2	3	2	155	
3	5	3	2	3	3	1210	
3	5	3	2	4	2	15	
3	5	3	2	4	3	40	
3	5	3	3	1	2	155	
3	5	3	3	1	3	1210	
3	5	3	3	2	2	155	
3	5	3	3	2	3	1210	
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3	5	3	3	3	3	1210	
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3	5	3	3	5	2	15	
3	5	3	3	5	3	40	
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3	5	3	4	1	3	121	
3	5	3	4	2	2	31	
3	5	3	4	2	3	121	
3	5	3	4	3	2	31	
3	5	3	4	3	3	121	
3	5	3	4	4	2	31	
3	5	3	4	4	3	121	
3	5	3	4	5	2	31	
3	5	3	4	5	3	121	
4	1	1	1	1	2	15	
4	1	1	1	1	3	40	40
4	1	2	0	1	2	7	
4	1	2	0	1	3	13	
4	1	2	1	1	2	140	
4	1	2	1	1	3	1170	
4	1	2	1	2	2	14	
4	1	2	1	2	3	39	
4	1	3	0	1	2	15	
4	1	3	0	1	3	40	
4	1	3	0	2	2	15	
4	1	3	0	2	3	40	
4	1	3	1	1	2	30	

a	b	i	j	t	q	Thm 4.24	Thm 4.22
4	1	3	1	1	3	120	
4	1	3	1	2	2	30	
4	1	3	1	2	3	120	
4	1	3	1	3	2	30	
4	1	3	1	3	3	120	
4	2	0	2	1	2	16	16
4	2	0	2	1	3	81	81
4	2	1	1	1	2	24	24
4	2	1	1	1	3	108	108
4	2	1	2	1	2	120	
4	2	1	2	1	3	1080	
4	2	1	2	2	2	15	
4	2	1	2	2	3	40	
4	2	2	0	1	2	7	
4	2	2	0	1	3	13	
4	2	2	1	1	2	140	
4	2	2	1	1	3	1170	
4	2	2	1	2	2	14	
4	2	2	1	2	3	39	
4	2	2	2	1	2	560	
4	2	2	2	1	3	10530	
4	2	2	2	2	2	560	
4	2	2	2	2	3	10530	
4	2	2	2	3	2	28	
4	2	2	2	3	3	117	
4	2	3	0	1	2	15	
4	2	3	0	1	3	40	
4	2	3	0	2	2	15	
4	2	3	0	2	3	40	
4	2	3	1	1	2	90	
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4	2	3	1	2	2	90	
4	2	3	1	2	3	480	
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4	2	3	2	1	3	360	
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4	2	3	2	2	3	360	
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4	2	4	1	2	3	4	
4	2	4	1	3	2	3	
4	2	4	1	3	3	4	
4	2	4	1	4	2	3	
4	2	4	1	4	3	4	
4	3	0	2	1	2	48	48
4	3	0	2	1	3	324	324
4	3	0	3	1	2	256	
4	3	0	3	1	3	6561	6561
4	3	0	3	2	2	16	
4	3	0	3	2	3	81	81

a	b	i	j	t	q	Thm 4.24	Thm 4.22
4	3	1	1	1	2	56	56
4	3	1	1	1	3	351	351
4	3	1	2	1	2	448	
4	3	1	2	1	3	9477	9477
4	3	1	2	2	2	24	
4	3	1	2	2	3	108	108
4	3	1	3	1	2	7680	
4	3	1	3	1	3	787320	
4	3	1	3	2	2	448	
4	3	1	3	2	3	9477	
4	3	1	3	3	2	24	
4	3	1	3	3	3	108	
4	3	2	0	1	2	7	
4	3	2	0	1	3	13	
4	3	2	1	1	2	196	
4	3	2	1	1	3	1521	
4	3	2	1	2	2	28	
4	3	2	1	2	3	117	117
4	3	2	2	1	2	3920	
4	3	2	2	1	3	136890	
4	3	2	2	2	2	560	
4	3	2	2	2	3	10530	
4	3	2	2	3	2	28	
4	3	2	2	3	3	117	
4	3	2	3	1	2	2240	
4	3	2	3	1	3	94770	
4	3	2	3	2	2	2240	
4	3	2	3	2	3	94770	
4	3	2	3	3	2	2240	
4	3	2	3	3	3	94770	
4	3	2	3	4	2	56	
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4	3	3	0	2	2	15	
4	3	3	0	2	3	40	
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4	3	3	1	2	3	1560	
4	3	3	1	3	2	30	
4	3	3	1	3	3	120	
4	3	3	2	1	2	420	
4	3	3	2	1	3	4680	
4	3	3	2	2	2	420	
4	3	3	2	2	3	4680	
4	3	3	2	3	2	420	
4	3	3	2	3	3	4680	
4	3	3	2	4	2	60	
4	3	3	2	4	3	360	
4	3	3	3	1	2	120	
4	3	3	3	1	3	1080	
4	3	3	3	2	2	120	
4	3	3	3	2	3	1080	
4	3	3	3	3	2	120	
4	3	3	3	3	3	1080	
4	3	3	3	4	2	120	

a	b	i	j	t	q	Thm 4.24	Thm 4.22
4	3	3	3	4	3	1080	
4	3	3	3	5	2	120	
4	3	3	3	5	3	1080	
4	3	4	1	1	2	7	
4	3	4	1	1	3	13	
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4	3	4	1	4	2	7	
4	3	4	1	4	3	13	
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4	3	4	2	1	3	13	
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4	3	4	2	2	3	13	
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4	3	4	2	3	3	13	
4	3	4	2	4	2	7	
4	3	4	2	4	3	13	
4	3	4	2	5	2	7	
4	3	4	2	5	3	13	
4	4	0	2	1	2	112	112
4	4	0	2	1	3	1053	1053
4	4	0	3	1	2	1792	1792
4	4	0	3	1	3	85293	85293
4	4	0	3	2	2	48	48
4	4	0	3	2	3	324	324
4	4	0	4	1	2	4096	
4	4	0	4	1	3	531441	
4	4	0	4	2	2	256	
4	4	0	4	2	3	6561	
4	4	0	4	3	2	16	
4	4	0	4	3	3	81	
4	4	1	1	1	2	120	120
4	4	1	1	1	3	1080	1080
4	4	1	2	1	2	2240	2240
4	4	1	2	1	3	94770	94770
4	4	1	2	2	2	56	56
4	4	1	2	2	3	351	351
4	4	1	3	1	2	7680	
4	4	1	3	1	3	787320	
4	4	1	3	2	2	448	
4	4	1	3	2	3	9477	
4	4	1	3	3	2	24	
4	4	1	3	3	3	108	
4	4	1	4	1	2	61440	
4	4	1	4	1	3	21257640	
4	4	1	4	2	2	61440	
4	4	1	4	2	3	21257640	
4	4	1	4	3	2	1792	
4	4	1	4	3	3	85293	
4	4	1	4	4	2	48	
4	4	1	4	4	3	324	
4	4	2	0	1	2	7	
4	4	2	0	1	3	13	
4	4	2	1	1	2	420	
4	4	2	1	1	3	4680	

a	b	i	j	t	q	Thm 4.24	Thm 4.22
4	4	2	1	2	2	60	60
4	4	2	1	2	3	360	360
4	4	2	2	1	2	3920	
4	4	2	2	1	3	136890	
4	4	2	2	2	2	560	
4	4	2	2	2	3	10530	
4	4	2	2	3	2	28	
4	4	2	2	3	3	117	
4	4	2	3	1	2	33600	
4	4	2	3	1	3	3790800	
4	4	2	3	2	2	33600	
4	4	2	3	2	3	3790800	
4	4	2	3	3	2	2240	
4	4	2	3	3	3	94770	
4	4	2	3	4	2	56	
4	4	2	3	4	3	351	
4	4	2	4	1	2	8960	
4	4	2	4	1	3	852930	
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4	4	2	4	5	2	112	
4	4	2	4	5	3	1053	
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4	4	3	0	1	3	40	
4	4	3	0	2	2	15	
4	4	3	0	2	3	40	
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4	4	3	1	1	3	4800	
4	4	3	1	2	2	450	
4	4	3	1	2	3	4800	
4	4	3	1	3	2	30	
4	4	3	1	3	3	120	
4	4	3	2	1	2	2100	
4	4	3	2	1	3	46800	
4	4	3	2	2	2	2100	
4	4	3	2	2	3	46800	
4	4	3	2	3	2	420	
4	4	3	2	3	3	4680	
4	4	3	2	4	2	60	
4	4	3	2	4	3	360	
4	4	3	3	1	2	1800	
4	4	3	3	1	3	43200	
4	4	3	3	2	2	1800	
4	4	3	3	2	3	43200	
4	4	3	3	3	2	1800	
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4	4	3	3	4	2	1800	
4	4	3	3	4	3	43200	
4	4	3	3	5	2	120	
4	4	3	3	5	3	1080	
4	4	3	4	1	2	240	
4	4	3	4	1	3	3240	
4	4	3	4	2	2	240	

a	b	i	j	t	q	Thm 4.24	Thm 4.22
4	4	3	4	2	3	3240	
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4	4	3	4	3	3	3240	
4	4	3	4	4	2	240	
4	4	3	4	4	3	3240	
4	4	3	4	5	2	240	
4	4	3	4	5	3	3240	
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4	4	4	1	1	3	40	
4	4	4	1	2	2	15	
4	4	4	1	2	3	40	
4	4	4	1	3	2	15	
4	4	4	1	3	3	40	
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4	4	4	3	3	3	40	
4	4	4	3	4	2	15	
4	4	4	3	4	3	40	
4	4	4	3	5	2	15	
4	4	4	3	5	3	40	
4	5	0	2	1	2	240	240
4	5	0	2	1	3	3240	3240
4	5	0	3	1	2	8960	8960
4	5	0	3	1	3	852930	852930
4	5	0	3	2	2	112	112
4	5	0	3	2	3	1053	1053
4	5	0	4	1	2	61440	
4	5	0	4	1	3	21257640	21257640
4	5	0	4	2	2	1792	
4	5	0	4	2	3	85293	85293
4	5	0	4	3	2	48	
4	5	0	4	3	3	324	324
4	5	0	5	1	2	1048576	
4	5	0	5	1	3	3486784401	
4	5	0	5	2	2	32768	
4	5	0	5	2	3	14348907	
4	5	0	5	3	2	1024	
4	5	0	5	3	3	59049	
4	5	0	5	4	2	32	
4	5	0	5	4	3	243	
4	5	1	1	1	2	248	248
4	5	1	1	1	3	3267	3267

a	b	i	j	t	q	Thm 4.24	Thm 4.22
4	5	1	2	1	2	9920	9920
4	5	1	2	1	3	882090	882090
4	5	1	2	2	2	120	120
4	5	1	2	2	3	1080	1080
4	5	1	3	1	2	79360	
4	5	1	3	1	3	23816430	23816430
4	5	1	3	2	2	2240	
4	5	1	3	2	3	94770	94770
4	5	1	3	3	2	56	
4	5	1	3	3	3	351	351
4	5	1	4	1	2	1904640	
4	5	1	4	1	3	2572174440	
4	5	1	4	2	2	61440	
4	5	1	4	2	3	21257640	
4	5	1	4	3	2	1792	
4	5	1	4	3	3	85293	
4	5	1	4	4	2	48	
4	5	1	4	4	3	324	
4	5	1	5	1	2	491520	
4	5	1	5	1	3	573956280	
4	5	1	5	2	2	491520	
4	5	1	5	2	3	573956280	
4	5	1	5	3	2	491520	
4	5	1	5	3	3	573956280	
4	5	1	5	4	2	7168	
4	5	1	5	4	3	767637	
4	5	1	5	5	2	96	
4	5	1	5	5	3	972	
4	5	2	0	1	2	7	
4	5	2	0	1	3	13	
4	5	2	1	1	2	868	
4	5	2	1	1	3	14157	
4	5	2	1	2	2	124	124
4	5	2	1	2	3	1089	1089
4	5	2	2	1	2	17360	
4	5	2	2	1	3	1274130	
4	5	2	2	2	2	2480	
4	5	2	2	2	3	98010	98010
4	5	2	2	3	2	60	
4	5	2	2	3	3	360	360
4	5	2	3	1	2	347200	
4	5	2	3	1	3	114671700	
4	5	2	3	2	2	33600	
4	5	2	3	2	3	3790800	
4	5	2	3	3	2	2240	
4	5	2	3	3	3	94770	
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4	5	2	4	1	2	277760	
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4	5	2	4	2	3	103204530	
4	5	2	4	3	2	277760	
4	5	2	4	3	3	103204530	
4	5	2	4	4	2	8960	
4	5	2	4	4	3	852930	
4	5	2	4	5	2	112	

a	b	i	j	t	q	Thm 4.24	Thm 4.22
4	5	2	4	5	3	1053	
4	5	2	5	1	2	35840	
4	5	2	5	1	3	7676370	
4	5	2	5	2	2	35840	
4	5	2	5	2	3	7676370	
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4	5	3	0	2	2	15	
4	5	3	0	2	3	40	
4	5	3	1	1	2	930	
4	5	3	1	1	3	14520	
4	5	3	1	2	2	930	
4	5	3	1	2	3	14520	
4	5	3	1	3	2	62	
4	5	3	1	3	3	363	363
4	5	3	2	1	2	9300	
4	5	3	2	1	3	435600	
4	5	3	2	2	2	9300	
4	5	3	2	2	3	435600	
4	5	3	2	3	2	900	
4	5	3	2	3	3	14400	
4	5	3	2	4	2	60	
4	5	3	2	4	3	360	
4	5	3	3	1	2	18600	
4	5	3	3	1	3	1306800	
4	5	3	3	2	2	18600	
4	5	3	3	2	3	1306800	
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4	5	3	3	3	3	1306800	
4	5	3	3	4	2	1800	
4	5	3	3	4	3	43200	
4	5	3	3	5	2	120	
4	5	3	3	5	3	1080	
4	5	3	4	1	2	7440	
4	5	3	4	1	3	392040	
4	5	3	4	2	2	7440	
4	5	3	4	2	3	392040	
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4	5	3	4	4	2	7440	
4	5	3	4	4	3	392040	
4	5	3	4	5	2	7440	
4	5	3	4	5	3	392040	
4	5	3	5	1	2	480	
4	5	3	5	1	3	9720	
4	5	3	5	2	2	480	
4	5	3	5	2	3	9720	
4	5	3	5	3	2	480	
4	5	3	5	3	3	9720	
4	5	3	5	4	2	480	
4	5	3	5	4	3	9720	

a	b	i	j	t	q	Thm 4.24	Thm 4.22
4	5	3	5	5	2	480	
4	5	3	5	5	3	9720	
4	5	4	1	1	2	31	
4	5	4	1	1	3	121	
4	5	4	1	2	2	31	
4	5	4	1	2	3	121	
4	5	4	1	3	2	31	
4	5	4	1	3	3	121	
4	5	4	1	4	2	31	
4	5	4	1	4	3	121	
4	5	4	2	1	2	155	
4	5	4	2	1	3	1210	
4	5	4	2	2	2	155	
4	5	4	2	2	3	1210	
4	5	4	2	3	2	155	
4	5	4	2	3	3	1210	
4	5	4	2	4	2	155	
4	5	4	2	4	3	1210	
4	5	4	2	5	2	15	
4	5	4	2	5	3	40	
4	5	4	3	1	2	155	
4	5	4	3	1	3	1210	
4	5	4	3	2	2	155	
4	5	4	3	2	3	1210	
4	5	4	3	3	2	155	
4	5	4	3	3	3	1210	
4	5	4	3	4	2	155	
4	5	4	3	4	3	1210	
4	5	4	3	5	2	155	
4	5	4	3	5	3	1210	
4	5	4	4	1	2	31	
4	5	4	4	1	3	121	
4	5	4	4	2	2	31	
4	5	4	4	2	3	121	
4	5	4	4	3	2	31	
4	5	4	4	3	3	121	
4	5	4	4	4	2	31	
4	5	4	4	4	3	121	
4	5	4	4	5	2	31	
4	5	4	4	5	3	121	
5	1	1	1	1	2	31	31
5	1	1	1	1	3	121	121
5	1	2	0	1	2	15	
5	1	2	0	1	3	40	40
5	1	2	1	1	2	155	
5	1	2	1	1	3	1210	
5	1	2	1	2	2	15	
5	1	2	1	2	3	40	
5	1	3	0	1	2	155	
5	1	3	0	1	3	1210	
5	1	3	0	2	2	15	
5	1	3	0	2	3	40	
5	1	3	1	1	2	620	
5	1	3	1	1	3	10890	
5	1	3	1	2	2	620	
5	1	3	1	2	3	10890	
5	1	3	1	3	2	30	

a	b	i	j	t	q	Thm 4.24	Thm 4.22
5	1	3	1	3	3	120	
5	1	4	0	1	2	31	
5	1	4	0	1	3	121	
5	1	4	0	2	2	31	
5	1	4	0	2	3	121	
5	1	4	0	3	2	31	
5	1	4	0	3	3	121	
5	1	4	1	1	2	62	
5	1	4	1	1	3	363	
5	1	4	1	2	2	62	
5	1	4	1	2	3	363	
5	1	4	1	3	2	62	
5	1	4	1	3	3	363	
5	1	4	1	4	2	62	
5	1	4	1	4	3	363	
5	2	0	2	1	2	32	32
5	2	0	2	1	3	243	243
5	2	1	1	1	2	48	48
5	2	1	1	1	3	324	324
5	2	1	2	1	2	496	
5	2	1	2	1	3	9801	9801
5	2	1	2	2	2	31	
5	2	1	2	2	3	121	121
5	2	2	0	1	2	15	
5	2	2	0	1	3	40	40
5	2	2	1	1	2	360	
5	2	2	1	1	3	4320	
5	2	2	1	2	2	24	
5	2	2	1	2	3	108	108
5	2	2	2	1	2	9920	
5	2	2	2	1	3	882090	
5	2	2	2	2	2	560	
5	2	2	2	2	3	10530	
5	2	2	2	3	2	28	
5	2	2	2	3	3	117	
5	2	3	0	1	2	155	
5	2	3	0	1	3	1210	
5	2	3	0	2	2	15	
5	2	3	0	2	3	40	
5	2	3	1	1	2	1860	
5	2	3	1	1	3	43560	
5	2	3	1	2	2	620	
5	2	3	1	2	3	10890	
5	2	3	1	3	2	30	
5	2	3	1	3	3	120	
5	2	3	2	1	2	2480	
5	2	3	2	1	3	98010	
5	2	3	2	2	2	2480	
5	2	3	2	2	3	98010	
5	2	3	2	3	2	2480	
5	2	3	2	3	3	98010	
5	2	3	2	4	2	60	
5	2	3	2	4	3	360	
5	2	4	0	1	2	31	
5	2	4	0	1	3	121	
5	2	4	0	2	2	31	
5	2	4	0	2	3	121	

a	b	i	j	t	q	Thm 4.24	Thm 4.22
5	2	4	0	3	2	31	
5	2	4	0	3	3	121	
5	2	4	1	1	2	186	
5	2	4	1	1	3	1452	
5	2	4	1	2	2	186	
5	2	4	1	2	3	1452	
5	2	4	1	3	2	186	
5	2	4	1	3	3	1452	
5	2	4	1	4	2	62	
5	2	4	1	4	3	363	
5	2	4	2	1	2	124	
5	2	4	2	1	3	1089	
5	2	4	2	2	2	124	
5	2	4	2	2	3	1089	
5	2	4	2	3	2	124	
5	2	4	2	3	3	1089	
5	2	4	2	4	2	124	
5	2	4	2	4	3	1089	
5	2	4	2	5	2	124	
5	2	4	2	5	3	1089	
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5	2	5	1	1	3	4	
5	2	5	1	2	2	3	
5	2	5	1	2	3	4	
5	2	5	1	3	2	3	
5	2	5	1	3	3	4	
5	2	5	1	4	2	3	
5	2	5	1	4	3	4	
5	2	5	1	5	2	3	
5	2	5	1	5	3	4	
5	3	0	2	1	2	96	96
5	3	0	2	1	3	972	972
5	3	0	3	1	2	1024	1024
5	3	0	3	1	3	59049	59049
5	3	0	3	2	2	32	32
5	3	0	3	2	3	243	243
5	3	1	1	1	2	112	112
5	3	1	1	1	3	1053	1053
5	3	1	2	1	2	1792	1792
5	3	1	2	1	3	85293	85293
5	3	1	2	2	2	48	48
5	3	1	2	2	3	324	324
5	3	1	3	1	2	7936	
5	3	1	3	1	3	793881	
5	3	1	3	2	2	496	
5	3	1	3	2	3	9801	
5	3	1	3	3	2	31	
5	3	1	3	3	3	121	
5	3	2	0	1	2	15	
5	3	2	0	1	3	40	40
5	3	2	1	1	2	840	
5	3	2	1	1	3	14040	
5	3	2	1	2	2	56	56
5	3	2	1	2	3	351	351
5	3	2	2	1	2	9920	
5	3	2	2	1	3	882090	
5	3	2	2	2	2	560	

a	b	i	j	t	q	Thm 4.24	Thm 4.22
5	3	2	2	2	3	10530	
5	3	2	2	3	2	28	
5	3	2	2	3	3	117	
5	3	2	3	1	2	79360	
5	3	2	3	1	3	23816430	
5	3	2	3	2	2	79360	
5	3	2	3	2	3	23816430	
5	3	2	3	3	2	2240	
5	3	2	3	3	3	94770	
5	3	2	3	4	2	56	
5	3	2	3	4	3	351	
5	3	3	0	1	2	155	
5	3	3	0	1	3	1210	
5	3	3	0	2	2	15	
5	3	3	0	2	3	40	
5	3	3	1	1	2	4340	
5	3	3	1	1	3	141570	
5	3	3	1	2	2	620	
5	3	3	1	2	3	10890	
5	3	3	1	3	2	30	
5	3	3	1	3	3	120	
5	3	3	2	1	2	17360	
5	3	3	2	1	3	1274130	
5	3	3	2	2	2	17360	
5	3	3	2	2	3	1274130	
5	3	3	2	3	2	2480	
5	3	3	2	3	3	98010	
5	3	3	2	4	2	60	
5	3	3	2	4	3	360	
5	3	3	3	1	2	9920	
5	3	3	3	1	3	882090	
5	3	3	3	2	2	9920	
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5	3	3	3	3	2	9920	
5	3	3	3	3	3	882090	
5	3	3	3	4	2	9920	
5	3	3	3	4	3	882090	
5	3	3	3	5	2	120	
5	3	3	3	5	3	1080	
5	3	4	0	1	2	31	
5	3	4	0	1	3	121	
5	3	4	0	2	2	31	
5	3	4	0	2	3	121	
5	3	4	0	3	2	31	
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5	3	4	1	1	3	4719	
5	3	4	1	2	2	434	
5	3	4	1	2	3	4719	
5	3	4	1	3	2	434	
5	3	4	1	3	3	4719	
5	3	4	1	4	2	62	
5	3	4	1	4	3	363	
5	3	4	2	1	2	868	
5	3	4	2	1	3	14157	
5	3	4	2	2	2	868	
5	3	4	2	2	3	14157	

a	b	i	j	t	q	Thm 4.24	Thm 4.22
5	3	4	2	3	2	868	
5	3	4	2	3	3	14157	
5	3	4	2	4	2	868	
5	3	4	2	4	3	14157	
5	3	4	2	5	2	124	
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5	3	4	3	2	2	248	
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5	3	4	3	3	2	248	
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5	3	4	3	4	3	3267	
5	3	4	3	5	2	248	
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5	3	5	1	2	3	13	
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5	3	5	1	4	3	13	
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5	3	5	2	3	2	7	
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5	3	5	2	4	2	7	
5	3	5	2	4	3	13	
5	3	5	2	5	2	7	
5	3	5	2	5	3	13	
5	4	0	2	1	2	224	224
5	4	0	2	1	3	3159	3159
5	4	0	3	1	2	7168	7168
5	4	0	3	1	3	767637	767637
5	4	0	3	2	2	96	96
5	4	0	3	2	3	972	972
5	4	0	4	1	2	32768	
5	4	0	4	1	3	14348907	14348907
5	4	0	4	2	2	1024	
5	4	0	4	2	3	59049	59049
5	4	0	4	3	2	32	
5	4	0	4	3	3	243	243
5	4	1	1	1	2	240	240
5	4	1	1	1	3	3240	3240
5	4	1	2	1	2	8960	8960
5	4	1	2	1	3	852930	852930
5	4	1	2	2	2	112	112
5	4	1	2	2	3	1053	1053
5	4	1	3	1	2	61440	
5	4	1	3	1	3	21257640	21257640
5	4	1	3	2	2	1792	

a	b	i	j	t	q	Thm 4.24	Thm 4.22
5	4	1	3	2	3	85293	85293
5	4	1	3	3	2	48	
5	4	1	3	3	3	324	324
5	4	1	4	1	2	2031616	
5	4	1	4	1	3	5208653241	
5	4	1	4	2	2	61440	
5	4	1	4	2	3	21257640	
5	4	1	4	3	2	1792	
5	4	1	4	3	3	85293	
5	4	1	4	4	2	48	
5	4	1	4	4	3	324	
5	4	2	0	1	2	15	
5	4	2	0	1	3	40	40
5	4	2	1	1	2	1800	
5	4	2	1	1	3	43200	
5	4	2	1	2	2	120	120
5	4	2	1	2	3	1080	1080
5	4	2	2	1	2	33600	
5	4	2	2	1	3	3790800	
5	4	2	2	2	2	2240	
5	4	2	2	2	3	94770	94770
5	4	2	2	3	2	56	
5	4	2	2	3	3	351	351
5	4	2	3	1	2	1190400	
5	4	2	3	1	3	952657200	
5	4	2	3	2	2	79360	
5	4	2	3	2	3	23816430	
5	4	2	3	3	2	2240	
5	4	2	3	3	3	94770	
5	4	2	3	4	2	56	
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5	4	2	4	2	2	634880	
5	4	2	4	2	3	643043610	
5	4	2	4	3	2	634880	
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5	4	3	0	1	2	155	
5	4	3	0	1	3	1210	
5	4	3	0	2	2	15	
5	4	3	0	2	3	40	
5	4	3	1	1	2	9300	
5	4	3	1	1	3	435600	
5	4	3	1	2	2	900	
5	4	3	1	2	3	14400	
5	4	3	1	3	2	60	
5	4	3	1	3	3	360	360
5	4	3	2	1	2	86800	
5	4	3	2	1	3	12741300	
5	4	3	2	2	2	17360	
5	4	3	2	2	3	1274130	
5	4	3	2	3	2	2480	
5	4	3	2	3	3	98010	

a	b	i	j	t	q	Thm 4.24	Thm 4.22
5	4	3	2	4	2	60	
5	4	3	2	4	3	360	
5	4	3	3	1	2	148800	
5	4	3	3	1	3	35283600	
5	4	3	3	2	2	148800	
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5	4	3	3	3	2	148800	
5	4	3	3	3	3	35283600	
5	4	3	3	4	2	9920	
5	4	3	3	4	3	882090	
5	4	3	3	5	2	120	
5	4	3	3	5	3	1080	
5	4	3	4	1	2	39680	
5	4	3	4	1	3	7938810	
5	4	3	4	2	2	39680	
5	4	3	4	2	3	7938810	
5	4	3	4	3	2	39680	
5	4	3	4	3	3	7938810	
5	4	3	4	4	2	39680	
5	4	3	4	4	3	7938810	
5	4	3	4	5	2	39680	
5	4	3	4	5	3	7938810	
5	4	4	0	1	2	31	
5	4	4	0	1	3	121	
5	4	4	0	2	2	31	
5	4	4	0	2	3	121	
5	4	4	0	3	2	31	
5	4	4	0	3	3	121	
5	4	4	1	1	2	930	
5	4	4	1	1	3	14520	
5	4	4	1	2	2	930	
5	4	4	1	2	3	14520	
5	4	4	1	3	2	930	
5	4	4	1	3	3	14520	
5	4	4	1	4	2	62	
5	4	4	1	4	3	363	
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5	4	4	2	5	2	124	
5	4	4	2	5	3	1089	
5	4	4	3	1	2	3720	
5	4	4	3	1	3	130680	
5	4	4	3	2	2	3720	
5	4	4	3	2	3	130680	
5	4	4	3	3	2	3720	
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5	4	4	3	4	2	3720	
5	4	4	3	4	3	130680	
5	4	4	3	5	2	3720	
5	4	4	3	5	3	130680	
5	4	4	4	1	2	496	

a	b	i	j	t	q	Thm 4.24	Thm 4.22
5	4	4	4	1	3	9801	
5	4	4	4	2	2	496	
5	4	4	4	2	3	9801	
5	4	4	4	3	2	496	
5	4	4	4	3	3	9801	
5	4	4	4	4	2	496	
5	4	4	4	4	3	9801	
5	4	4	4	5	2	496	
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5	4	5	2	3	3	130	
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5	4	5	3	5	2	15	
5	4	5	3	5	3	40	
5	5	0	2	1	2	480	480
5	5	0	2	1	3	9720	9720
5	5	0	3	1	2	35840	35840
5	5	0	3	1	3	7676370	7676370
5	5	0	3	2	2	224	224
5	5	0	3	2	3	3159	3159
5	5	0	4	1	2	491520	491520
5	5	0	4	1	3	573956280	573956280
5	5	0	4	2	2	7168	7168
5	5	0	4	2	3	767637	767637
5	5	0	4	3	2	96	96
5	5	0	4	3	3	972	972
5	5	0	5	1	2	1048576	
5	5	0	5	1	3	3486784401	
5	5	0	5	2	2	32768	
5	5	0	5	2	3	14348907	
5	5	0	5	3	2	1024	
5	5	0	5	3	3	59049	

a	b	i	j	t	q	Thm 4.24	Thm 4.22
5	5	0	5	4	2	32	
5	5	0	5	4	3	243	
5	5	1	1	1	2	496	496
5	5	1	1	1	3	9801	9801
5	5	1	2	1	2	39680	39680
5	5	1	2	1	3	7938810	7938810
5	5	1	2	2	2	240	240
5	5	1	2	2	3	3240	3240
5	5	1	3	1	2	634880	634880
5	5	1	3	1	3	643043610	643043610
5	5	1	3	2	2	8960	8960
5	5	1	3	2	3	852930	852930
5	5	1	3	3	2	112	112
5	5	1	3	3	3	1053	1053
5	5	1	4	1	2	2031616	
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5	5	2	1	1	2	3720	
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5	5	2	1	2	3	3267	3267
5	5	2	2	1	2	148800	
5	5	2	2	1	3	35283600	
5	5	2	2	2	2	9920	9920
5	5	2	2	2	3	882090	882090
5	5	2	2	3	2	120	120
5	5	2	2	3	3	1080	1080
5	5	2	3	1	2	1190400	
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5	5	2	3	2	2	79360	
5	5	2	3	2	3	23816430	
5	5	2	3	3	2	2240	
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5	5	2	4	2	2	19681280	
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a	b	i	j	t	q	Thm 4.24	Thm 4.22
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5	5	3	1	2	3	43560	
5	5	3	1	3	2	124	124
5	5	3	1	3	3	1089	1089
5	5	3	2	1	2	384400	
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5	5	3	5	1	3	71449290	
5	5	3	5	2	2	158720	
5	5	3	5	2	3	71449290	

a	b	i	j	t	q	Thm 4.24	Thm 4.22
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5	5	4	1	4	3	363	
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5	5	4	2	4	2	1860	
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5	5	4	5	3	2	992	
5	5	4	5	3	3	29403	
5	5	4	5	4	2	992	

a	b	i	j	t	q	Thm 4.24	Thm 4.22
5	5	4	5	4	3	29403	
5	5	4	5	5	2	992	
5	5	4	5	5	3	29403	
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5	5	5	1	1	3	121	
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5	5	5	4	3	2	31	
5	5	5	4	3	3	121	
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5	5	5	4	4	3	121	
5	5	5	4	5	2	31	
5	5	5	4	5	3	121	

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